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THE UNSTEADY FLOW FIELD ABOUT A
RIGHT CIRCULAR CONE IN UNSTEADY FLIGHT
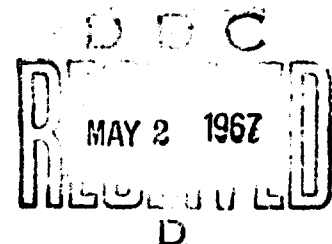
E. A. BRONG

GENERAL ELECTRIC COMPANY

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THE UNSTEADY FLOW FIELD ABOUT A
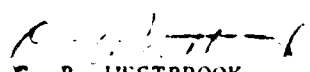# RIGHT CIRCULAR CONE IN UNSTEADY FLIGHT

E. A. BRONG

GENERAL ELECTRIC COMPANY

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FOREWORD

The information released in this report has been generated in support of studies on the Hypersonic Dynamic Stability Characteristics of Lifting and Non-Lifting Reentry Vehicles. These studies were conducted by the General Electric Company, Reentry Systems Department, for the Stability and Control Section of the Flight Dynamics Laboratory of the Air Force Research and Technology Division. The program was sponsored under Air Force Contract Number AF 33(657)-11411, Project Number 8219 and Task Number 821902. Mr. J. Jenkins of The Control Criteria Branch, RTD, is the project engineer on the contract. The project supervisor for the General Electric Company was Mr. L. A. Marshall.

This technical report has been reviewed and is approved.

C. B. WESTBROOK
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Flight Control Division
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ABSTRACT

The inviscid flow field about a right circular cone in unsteady planar flight is analyzed by a perturbation technique which is an extension of Stone's treatment of the cone at small yaw. A solution is found in the form of infinite series in the time rates of change of the pitch rate and angle of attack. The linear stability derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$ as well as "higher order" stability derivatives such as $C_{m_{\dot{q}}}$ and $C_{m_{\ddot{\alpha}}}$ are presented for a wide range of cone angles and Mach numbers.

The stability derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$, as obtained from this solution are compared to results obtained from second order potential theory, Newtonian impact theory, and an unsteady flow theory due to Zartarian, Hsu, and Ashley. Both the potential theory and the impact theory predict that $C_{m_{\dot{\alpha}}}$ rapidly approaches zero at high Mach numbers while the present theory indicates that $C_{m_{\dot{\alpha}}}$ approaches a value which is on the order of 10 percent to 20 percent of C_{m_q}.

Numerical results obtained from the present theory are also compared to ground-test data on $C_{m_q} + C_{m_{\dot{\alpha}}}$. The agreement is found to be generally good, although the data in some instances indicate a pronounced Reynolds number effect.

The numerical results for the "higher order" coefficients are used to predict the effect of reduced frequency on the parameter $C_{m_q} + C_{m_{\dot{\alpha}}}$ as obtained by the forced

oscillation testing techniques. It is found that the predicted effect is very small over the range of reduced frequencies likely to be encountered.

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SYMBOLS

A, B   Coefficients defined in Equations (5.7) and (5.8)

A_B Body base area

C_m Moment Coefficients, Moment $\div {}^\pi/8\rho_\infty \ U_\infty^2 \ D^2 L$

C_N Normal Force Coefficient, Normal Force $\div {}^\pi/8\rho_\infty U_\infty^2 D^2$

$C_{m\dot\alpha}$ $\left(\dfrac{2\,U_\infty}{L}\right) \dfrac{\partial C_m}{\partial \dot\alpha}$

C_{m_q} $\left(\dfrac{2\,U_\infty}{L}\right) \dfrac{\partial C_m}{\partial q}$

C_{N_α} $\left(\dfrac{\delta C_N}{\delta \alpha}\right)$

$C_{N_{\dot\alpha}}$ $\left(\dfrac{2\,U_\infty}{L}\right) \dfrac{\partial C_N}{\partial \dot\alpha}$

$C_{N_{\ddot\alpha}}$ $\left(\dfrac{2\,U_\infty^2}{L}\right) \dfrac{\partial C_N}{\partial \ddot\alpha}$

$C_{N_{\dddot\alpha}}$ $\left(\dfrac{2\,U_\infty^3}{L}\right) \dfrac{\partial C_N}{\partial \dddot\alpha}$

C_{N_q} $\left(\dfrac{2\,U_\infty}{L}\right) \dfrac{\delta C_N}{\partial q}$

$C_{N_{q_o}}$ C_{N_q} for $\dfrac{x_{cg}}{L} = 0$

$C_{N_{\dot q}}$ $\left(\dfrac{2\,U_\infty^2}{L}\right) \dfrac{\partial C_N}{\partial \dot q}$

$C_{N_{\ddot q}}$ $\left(\dfrac{2\,U_\infty^3}{L}\right) \dfrac{\partial C_N}{\partial \ddot q}$

c	Speed of sound $\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}$
D	Base Diameter
e^2	$\frac{\partial p}{\partial S}\Big|_\rho$
f	Frequency of oscillation
F_s	Function defining the shock shape
H	Stagnation enthalpy
h	Static enthalpy
j	Index referring to angle of attack variation (j = 1) or pitch rate variation (j = 2)
L	Body length
M	Mach number
n	Subscript relating to the order of derivative of α or q
$\vec{n}_B$	Unit outward normal to the body surface
$\vec{n}_s$	Unit inward normal to the shock
p	Pressure
q	Pitch rate
q_∞	$1/2\, \rho_\infty\, U_\infty^2$ – dynamic pressure
R, ω, φ	Spherical coordinates
R	Universal gas constant
$\vec{R}, \vec{\omega}, \vec{\varphi}$	Unit vectors of the spherical coordinate system
S	Specific Entropy
t	Time
T	Temperature

U_∞	Vehicle speed
u, v, w	Velocity components along the R, ω, φ directions
$\vec{V}$	Velocity of the fluid relative to the body
V_n	Fluid speed normal to the shock
$\vec{V}_{cg}$	Velocity of the center of gravity
x, y, z	Cartesian coordinates
$\vec{x}, \vec{y}, \vec{z}$	Unit vectors of the Cartesian System
x_{cg}	Location of the center of gravity (aft of the vertex)
α	Angle of attack
γ	Ratio of specific heats
$\gamma *$	$\dfrac{\rho c^2}{p}$
$\delta_{i,j}$	Kronecker delta
ϵ	Perturbation parameter
ρ	Density
$\vec{\Omega}$	$\nabla \times \vec{V} = \text{Curl } \vec{V}$

Superscripts

($\cdot$)	Derivative with respect to time

Subscripts

∞	In the free stream
0	Steady state, zero yaw condition
s	Pertaining to the downstream side of the shock
B	Pertaining to the body

1. INTRODUCTION

As a re-entry vehicle penetrates the atmosphere, its pitch rate (q) and angle of attack (α) vary in an oscillatory manner and cause the flow field about the vehicle to be in an unsteady state. For most purposes, q and the rates of change of q, and α are small enough that the differences between the unsteady field and a quasi-steady field may be considered negligible. An exception to this rule occurs in the evaluation of aerodynamic forces and moments. The small contributions of unsteady effects to the normal force and pitching moment appear as damping coefficients in the equations governing the rigid body motion of the vehicle and play an important role in determining the loads the vehicle must withstand.

In this work, the unsteady flow field about a vehicle (specifically, a right circular cone) is examined under the assumption that the motion of the vehicle is given. The vehicle motion appears explicitly in the equations and boundary conditions for determining the flow field in a body-fixed coordinate system (Section 4.1). In order to make the boundary value problem tractable, perturbation parameters which are measures of the departure of the flow field from a steady-state, axis-symmetric field are introduced (Equation 5.14) and used to effect a linearization of the problem (Section 5.2). The first-order effects of these parameters are considered, and a formal solution in the form of an infinite series which gives the flow field in terms of the vehicle motion, is assumed (Equation 5.26). Thus the mathematical problem is reduced to one of solving an infinite sequence of linear, ordinary differential equations (Equations 5.27) with variable coefficients which are determined by the steady field at zero yaw. These

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equations are solved by numerical methods (Section 6). The numerical results, when properly combined (Appendix I), give the unsteady flow field to the first order of the perturbation parameters. The unsteady flow effects on the normal force and pitching moment coefficients are obtained by integration of the surface pressures (Appendix III).

Results obtained by the method presented here are compared to theoretical results from other methods of varying degrees of approximation, and are also compared to experimental results (Section 7). From the cases examined, it does not appear that there is an orderly pattern of agreement between the results obtained here and the potential theory of Tobak and Wehrend (Reference 13), the shock expansion theory of Zartarian, Hsu and Ashley (Reference 14), or Newtonian impact theory. Comparisons with experimental data do show fair agreement.

2. ASSUMPTIONS AND RESTRICTIONS

The flow field is determined by the body geometry, the nature of the gas, and the flight conditions. The assumptions and restrictions pertaining to each are tabulated below:

1. Body Geometry - The body is assumed to be a right circular cone.

2. Nature of the Gas - The gas is assumed to be inviscid, non-conducting and at chemical and thermodynamic equilibrium. It is represented analytically as a γ^* gas as described in Reference 1. This includes an ideal gas with constant specific heats as a special case.

3. Flight Conditions - The vehicle trajectory is assumed to be planar with three degrees of freedom - i.e., two-degrees-of-freedom in translation in the plane of the trajectory and one-degree-of-freedom in rotation normal to the place of the trajectory. It is further assumed that the speed of the vehicle is constant and sufficiently high that the flow about the vehicle is supersonic with respect to the vehicle. This reduces the number of degrees-of-freedom to two-angle of attack and pitch. Only first order effects of these on the flow field are considered.

The variation of ambient pressure and density along the trajectory is neglected in the analysis.

3. DESCRIPTION OF THE PERTURBATION SCHEME

The flow field about a body in flight is determined by the solution to the non-linear boundary value problem given in Section 4.1. This boundary value problem is stated in a coordinate system fixed in the body and consequently the motions of the body $\vec{V}_{cg}$ and $\vec{\Omega}$, appear as "driving functions" in the problem. For the simple planar trajectory considered here the motions of the body are given in terms of two functions of time, $\alpha(t)$ and $q(t)$, and the constant speed $U_\infty = \left| \vec{V}_{cg} \right|$, by Equation (4.13). The flow field variables are functionals of the functions $\alpha(t)$ and $q(t)$ in that they depend on all the values taken on by $\alpha(t)$ and $q(t)$ in the interval from the initial time, to the current time, t, and are ordinary functions of position as given by the three coordinates, (R, ω, φ).

In the perturbation scheme utilized in this work, $\alpha(t)$ and $\dfrac{q(t)\, L}{U_\infty}$ are taken to be small quantities on the order of ϵ_1 and ϵ_2, respectively, which then become the perturbation parameters with the substitutions:

$$\alpha(t) = \epsilon_1\, \bar{\alpha}(t)$$

$$q(t) = \epsilon_2\, \bar{q}(t)\, \frac{U_\infty}{L}$$

the pressure, for example, will be a functional of the functions $\alpha(t)$ and $q(t)$ and an ordinary function of the parameters ϵ_1 and ϵ_2. It is assumed that p can be expanded in Taylor's series in the parameters, ϵ_1 and ϵ_2, to give a series of the form

$$p = p_0 + \epsilon_1 p_1 + \epsilon_2 p_2 + \dots \text{(H.O.T.)}$$

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The coefficient, p_0, is the pressure field produced by the body in steady flight at zero angle of attack and can be found by established methods. The coefficients, p_1 and p_2, give the first order effects of angle of attack and pitching rate, respectively, and are to be determined by solution of Equations (4.16) and (4.17). They are functionals of the functions $\bar{\alpha}(t)$ and $\bar{q}(t)$ and ordinary functions of the spatial coordinates. In Section 4.2 it is shown that they can be represented formally by series of the type:

$$p_1 = \sum_{n=0}^{\infty} p_{1,n} \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{dt^n} \cos \varphi = \left\{ p_{1,0} \; \bar{\alpha} + p_{1,1} \left(\frac{R}{U_\infty}\right) \frac{d\bar{\alpha}}{dt} + \cdots \right\} \cos \varphi$$

(and a similar series for p_2 as given in Appendix I) where the $p_{1,n}$ are functions only of the ray angle, ω. This solution holds after "starting transients" have died out. The coefficient, $p_{1,0}$, gives the effect of small yaw in the steady state. The coefficients, $p_{1,1}$, $p_{1,2}$, etc., give the effects of time varying angle of attack.

It is the coefficients, $p_{j,n}$ (and corresponding quantities for the other flow field variables), which are found as a result of this analysis. The method of finding them is numerical and is described in Sections 4 and 5. The pressure coefficients yield corresponding forces and moment coefficients (Appendix III) which are static and dynamic stability derivatives.

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4. DERIVATION OF THE PERTURBATION EQUATIONS

4.1 STATEMENT OF THE BOUNDARY VALUE PROBLEM IN GENERAL FORM

The inviscid flow field boundary value problem can be stated to an observer in a body-fixed coordinate system (x, y, z), Figure 1, by a transformation of coordinates from an inertial system. Lamb, (Reference 3), gives the appropriate transformed continuity and Euler Equations:

the continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0; \tag{4.1}$$

the Euler Equations:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} + 2 (\vec{\Omega} \times \vec{V}) + \frac{\nabla p}{\rho} = -\frac{d\vec{V}_{cg}}{dt} - \vec{\Omega} \times \vec{V}_{cg} + \vec{r} \times \frac{d\vec{\Omega}}{dt} + (\vec{\Omega} \times \vec{r}) \times \vec{\Omega}. \tag{4.2}$$

The form of the continuity equation is unaltered by the transformation. The form of the Euler equations is altered in that the acceleration of a fixed point in the moving frame of reference appears as a body force (per unit mass) involving the vector velocity of the center of gravity, $\vec{V}_{cg}(t)$, and the vector rotation of the body, $\vec{\Omega}(t)$, on the right hand side. The vector rotation causes a given fluid particle to experience a Coriolis acceleration, $2 (\vec{\Omega} \times \vec{V})$, which is present on the left hand side. The derivatives, $\frac{d\vec{V}_{cg}}{dt}$ and $\frac{d\vec{\Omega}}{dt}$, appearing in equation (4.2) are those observed in the body-fixed coordinates. For example, $\frac{d\vec{V}_{cg}}{dt}$ is

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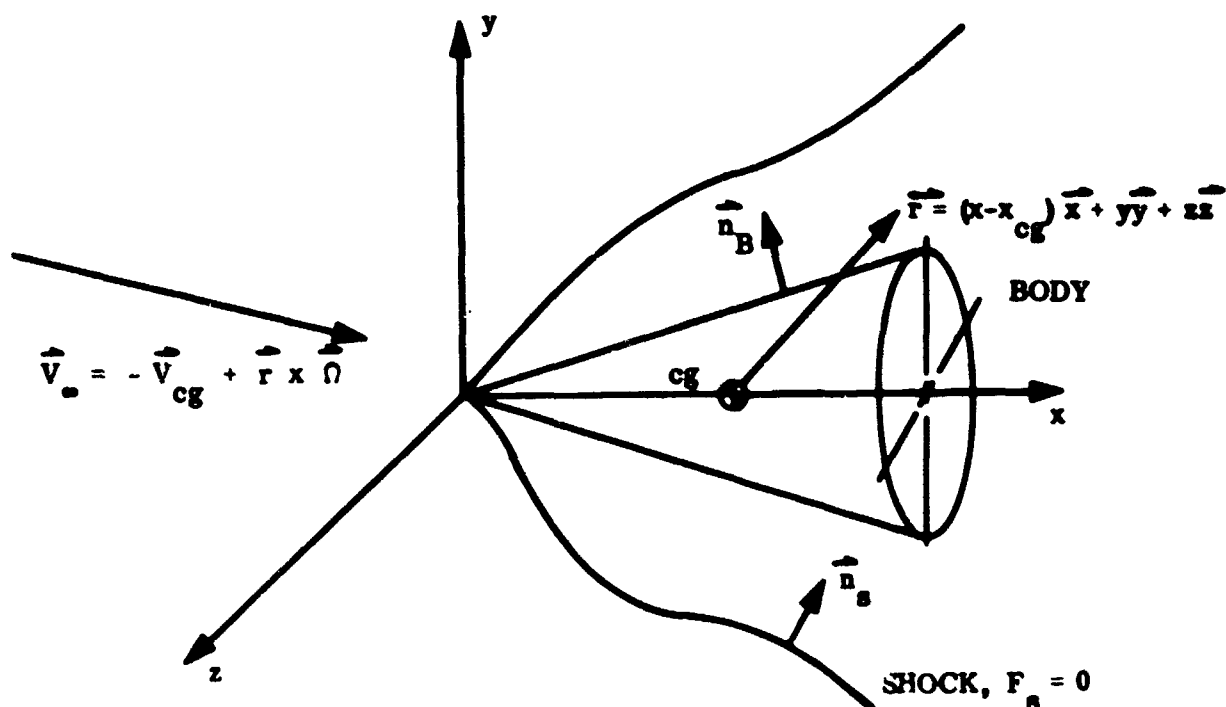
Figure 1. Inviscid Flow Field Boundary Value Problem

$$\frac{d\vec{V}_{cg}}{dt} = \vec{x}\ \frac{d\ (\vec{x}\ \cdot\ \vec{V}_{cg})}{dt}\ +\ \vec{y}\ \frac{d\ (\vec{y}\ \cdot\ \vec{V}_{cg})}{dt}\ +\ \vec{z}\ \frac{d\ (\vec{z}\ \cdot\ \vec{V}_{cg})}{dt}$$

The continuity and Euler equations must be complemented by the energy equation for the adiabatic flow:

$$\frac{\partial S}{\partial t}\ +\ \vec{V}\ \cdot \nabla S = 0 \tag{4.3}$$

and an equation of state for the gas, here taken to be:

$$p = p\ (\rho,\ s) \tag{4.4}$$

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The function indicated in Equation (4.4) depends on whether the gas is an ideal gas, a gas at chemical equilibrium, or a gas in some "frozen" composition. The development of the equations can, however, proceed without specification of the precise form of the equation of state.

Equations (4.1) through (4.4) are a complete set of flow equations for the determination of the pressure, p, the density, ρ, the entropy, S, and the three components of the fluid velocity (measured in the moving frame of reference), $\vec{V}$, for prescribed motions of the body, $\vec{V}_{cg}$ and $\vec{\Omega}$. In order to solve them, initial conditions at some instant of time and boundary conditions at the body and shock must be given.

As initial conditions it is assumed that at $t = 0$, the field is the steady-state, axi-symmetric field produced by a uniform forward translation at speed U_∞ of the body along its axis of symmetry.

At the body surface, the flow must be tangent to the body surface and the boundary condition is:

$$\vec{V} \cdot \vec{n}_B = 0 \tag{4.5}$$

On the shock surface:

$$F_s (x, \, , z, t) = 0, \tag{4.6}$$

(which must be found as a part of the solution) the shock equations give the flow variables as functions of the relative velocity of the shock and the free stream flow, and the in-stantaneous unit inward normal to the shock.

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$$\vec{n}_s = \pm \frac{\nabla F_s}{|\nabla F_s|} \qquad (4.7)$$

The sign in equation (4.7) is to be chosen so that $\vec{\omega} \cdot \vec{n}_s$ is negative.

The shock equations can be reduced to the following set of three algebraic equations:

$$\left. \begin{array}{l} \rho_s\, V_{ns} = \rho_\infty\, V_{n\infty} \\[2ex] \rho_s\, V_{ns}^2 + p_s = \rho_\infty\, V_{n\infty}^2 + p_\infty \\[2ex] h\,(p_s,\, \rho_s) + \dfrac{V_{ns}^2}{2} = h\,(p_\infty \rho_\infty) + \dfrac{V_{n\infty}^2}{2} \end{array} \right\} \qquad (4.8)$$

These equations express, respectively, the conservation of mass, momentum normal to the shock, and energy in a form which utilizes the result that the velocity component tangent to the shock is unchanged in crossing the shock. In these equations $V_{n\infty}$ is the component normal to the shock, of the relative velocity between the shock and the flow on the upstream side of the shock, and V_{ns} is the corresponding component on the downstream side of the shock.

The flow velocity on the upstream side of the shock is given by:

$$\vec{V}_\infty = -\vec{V}_{cg} + (\vec{r} \times \vec{\omega}) \qquad (4.9)$$

and the component of shock velocity along its normal is given by (reference 3, page 7):

$$-\frac{\dfrac{\partial F_s}{\partial t}}{\left| \nabla F_s \right|}\ .$$

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Therefore, V_{n1}, is given by:

$$V_{n1} = \vec{V}_\infty \cdot \vec{n}_s + \frac{\dfrac{\partial F_s}{\partial t}}{\left| \nabla F_s \right|} \qquad (4.10)$$

$$= -\vec{V}_{cg} \cdot \vec{n}_s + (\vec{r} \times \vec{\Omega}) \cdot \vec{n}_s + \frac{\dfrac{\partial F_s}{\partial t}}{\left| \nabla F_s \right|}$$

Equations (4.8) are presumed to be solved in the form*:

$$\left.\begin{array}{l} p_s = p_s \, (V_{n\infty}, \, p_\infty, \, \rho_\infty) \\[3mm] \rho_s = \rho_s \, (V_{n\infty}, \, p_\infty, \, \rho_\infty) \\[3mm] V_{ns} - V_{n\infty} \equiv \Delta V_n = \Delta V_n \, (V_{n\infty}, \, p_\infty, \, \rho_\infty) \end{array}\right\} \qquad (4.11)$$

The first two of these equations give the pressure and density downstream of the shock explicitly in terms of the function giving the shock, F_s, and the motions of the body, $\vec{V}_{cg}$ and $\vec{\Omega}$ by use of equation (4.10). The third equation gives the three velocity components of the flow on the downstream side of the shock by further use of the equation:

$$\vec{V}_s = -\vec{V}_{cg} + (\vec{r} \times \vec{\Omega}) + \Delta V_n \, \vec{n}_s \qquad (4.12)$$

4.2 REDUCTION OF THE BOUNDARY VALUE PROBLEM TO PERTURBATION FORM

The equations and conditions of Section 4.1 define the boundary value problem for determination of the inviscid flow field about a body whose motions, $\vec{V}_{cg}$ and $\vec{\Omega}$, are

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*There are numerous procedures in the literature for obtaining such a solution numerically. Reference 4 presents the procedure used in the subsequent numerical work.

known. In this Section, the boundary value problem is reduced to a perturbation form applicable to right circular cones which move at constant speed, $\left|\vec{V}_{cg}\right|$ = const. = U_∞, in such a manner that the z-axis is always parallel to a fixed reference direction.**

For the type of motion just described, the vector velocity and rotation can be written in terms of two functions, α (t) and q (t):

$$\vec{V}_{cg} = U_\infty (-\vec{x} \cos \alpha + \vec{y} \sin \alpha)$$

$$\vec{\Omega} = q \vec{z}$$

(4.13)

To put the problem in perturbation form it is assumed that $\alpha \ll 1$ and $\dfrac{q\,L}{U_\infty} \ll 1$, and the parameters ϵ_1 and ϵ_2 which measure the magnitudes of α and $\dfrac{qL}{U_\infty}$, are introduced by

$$\alpha = \epsilon_1 \bar{\alpha}$$

and $\dfrac{qL}{U_\infty} = \epsilon_2 \bar{q}$

(4.14)

In Equation (4.14), $\bar{\alpha}$ (t) and $\bar{q}$ (t) are assumed to be of the order of one in the time interval of interest, and ϵ_1 and ϵ_2 (which are constants) are assumed to be much less than one. The flow field variables can be regarded as functionals of the functions α (t) q (t) (in the sense defined in Reference 3) and ordinary functions of the perturbation parameters ϵ_1 and ϵ_2.

**Although this analysis does not employ stability axes the force and moment derivatives obtained are conventional.

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In the perturbation scheme, it is assumed that each flow field variable can be expanded in Taylor's Series in the variables ϵ_1 and ϵ_2. For example, it is assumed that the pressure can be written in the form:

$$p = p_0(x,y,z) + \epsilon_1 p_1(x,y,z,t) + \epsilon_2 p_2(x,y,z,t) + \ldots \text{ H.O.T.}. \tag{4.15}$$

These expanded forms are then substituted into the equations defining the boundary value problem and the resulting set of perturbed equations is separated into a number of sets of equations for the perturbations by the usual process of equating forms in like powers of ϵ_1 and ϵ_2. (Only the three lowest order terms are considered.)

The boundary value problem so defined for the lowest order coefficients, p_0, ρ_0, S_0 and $\vec{V}_0$, is, of course, the problem of determining the steady-state, axisymmetric field produced by the body as it translates with constant speed, U_∞, parallel to its axis of symmetry. Under the assumptions listed in Section 2, this field will also be conical and can be solved by the methods of Taylor and Maccoll (Reference 5), or Romig (Reference 6). In the subsequent work it is assumed that this field is known*. The equations defining the steady state field are given in Appendix II.

Substitution of Equation (4.15) and like expressions for the density, entropy, and fluid velocity into Equations (4.1) through (4.4) to obtain the two sets of equations governing the coefficients of ϵ_1 and ϵ_2 yields.

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*The numerical procedure used to obtain it is given in Reference 7.

$$\frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_o \vec{V}_j + \rho_j \vec{V}_o) = 0;$$

$$\frac{\partial \vec{V}_j}{\partial t} + (\vec{V}_o \cdot \nabla)\, \vec{V}_j + (\vec{V}_j \cdot \nabla)\, \vec{V}_o + \frac{\nabla p_j}{\rho_o} - \frac{\nabla p_o}{\rho_o^2}\, \rho_j = \vec{F}_j;$$

$$\frac{\partial S_j}{\partial t} + \vec{V}_o \cdot \nabla S_j + \vec{V}_j \cdot \nabla S_o = 0 \;;$$

$$p_j = c_o^{\,2}\, \rho_j + e_o^2\, S_j,$$

$$j = 1, 2 \qquad (4.16)$$

where:

$$\vec{F}_j = \begin{cases} - U_\infty\, \dfrac{d\,\vec{\alpha}}{dt}\, \vec{y}\,, & j = 1 \\[2em] 2(\vec{V}_o \times \vec{z})\, \bar{q}\, \dfrac{U_\infty}{L} + \dfrac{U_\infty^{\,2}}{L}\, \bar{q}\vec{y} + (\vec{r} \times \vec{z})\, \dfrac{U_\infty}{L}\, \dfrac{d\,\bar{q}}{dt}\,, & j = 2\,. \end{cases} \qquad (4.17)$$

The boundary condition at the body surface, Equation (4.5), separates into the two sets of conditions:

$$\vec{V}_j \cdot \vec{n}_B = 0, \;\; (j = 1, 2) \qquad (4.18)$$

For the conical geometry, the equation giving the shock, Equation (4.6), is most conveniently expressed in terms of the spherical coordinates, (R, ω, φ), Figure 2, in a form giving the ray angle, ω, as an explicit function of the radial coordinate, R, meridional angle, φ, and time, t:

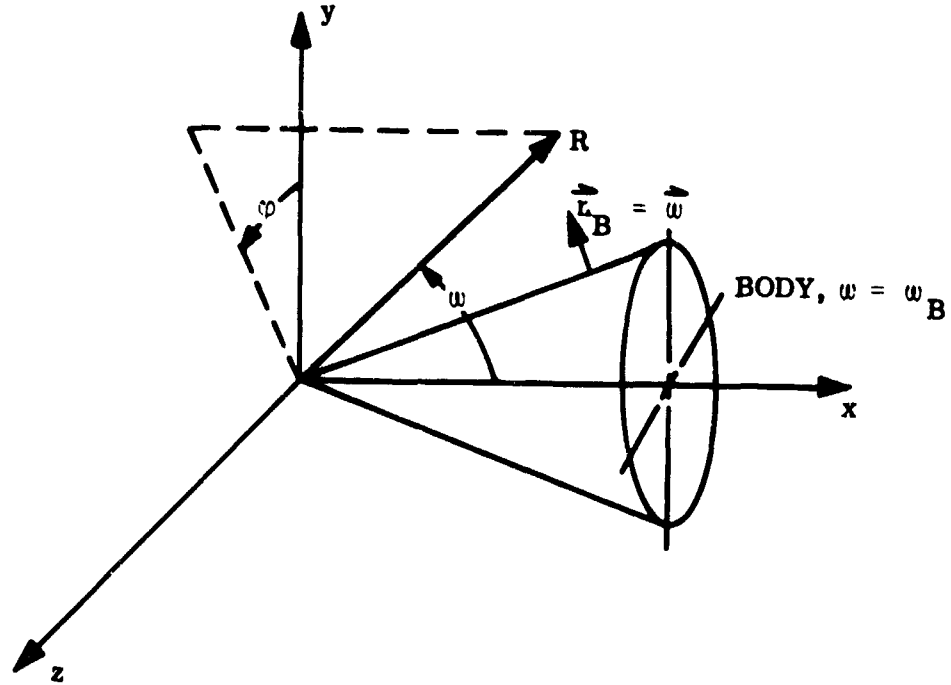$$F_s \equiv \omega_s\,(R, \varphi, t) - \omega = 0 \qquad (4.19)$$

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Figure 2. The Spherical Coordinate System (R, ω, φ)

The perturbation form for the shock is then (to the three lowest order terms):

$$\omega = \omega'_s = \omega'_{so} + \epsilon_1 \, \omega'_{s1} \, (R, \varphi, t) + \epsilon_2 \, \omega_{s2} \, (R, \varphi, t) \tag{4.20}$$

where ω'_{so} is the constant shock angle from the steady-state, axisymmetric, conical flow solutions. The unit inward normal to the shock (Equation 4.7) becomes:

$$\vec{n}_s = -\vec{\omega} + \sum_{j=1}^{2} \epsilon_j \left\{ \vec{R} \; R \frac{\partial \omega_{sj}}{\partial R} + \frac{\vec{\varphi}}{\sin \omega_{so}} \frac{\partial \omega_{sj}}{\partial \varphi} \right\} \tag{4.21}$$

and the flow velocity on the upstream side of the perturbed shock is:

$$\vec{V}_\infty = \vec{R} \left\{ u_{\infty o} + \epsilon_1 u_{\infty 1} + \epsilon_2 u_{\infty 2} \right\} + \vec{\omega} \left\{ v_{\infty o} + \epsilon_1 v_{\infty 1} + \epsilon_2 v_{\infty 2} \right\} + \vec{\varphi} \left\{ \epsilon_1 w_{\infty 1} + \epsilon_2 w_{\infty 2} \right\} \tag{4.22}$$

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where:

$$u_{\infty 0} = U_{\infty} \cos \omega_{so}$$

$$u_{\infty 1} = -U_{\infty}\left(\bar{\alpha} \sin \omega_{so} \cos \varphi + \omega_{s1} \sin \omega_{so} \right)$$

$$u_{\infty 2} = \frac{\bar{q} U_{\infty}}{L} x_{cg} \sin \omega_{so} \cos \varphi - U_{\infty} \omega_{s2} \sin \omega_{so} \qquad (4.23)$$

$$v_{\infty 0} = - U_{\infty} \sin \omega_{so}$$

$$v_{\infty 1} = - U_{\infty}(\bar{\alpha} \cos \omega_{so} \cos \varphi + \omega_{s1} \cos \omega_{so})$$

$$v_{\infty 2} = \frac{\bar{q} U_{\infty}}{L} (x_{cg} \cos \omega_{so} - R) \cos \varphi - U_{\infty} \omega_{s2} \cos \omega_{so}$$

$$w_{\infty 1} = \bar{\alpha} U_{\infty} \sin \varphi$$

$$w_{\infty 2} = \frac{\bar{q} U_{\infty}}{L}\left(R \cos \omega_{so} - x_{cg}\right) \sin \varphi$$

The perturbation form for the normal component of relative velocity between the upstream flow and the shock is obtained from Equation (4.10) as:

$$V_{n\infty} = - v_{\infty 0} + \sum_{j=1}^{2} \epsilon_j \bar{V}_{nj} \qquad (4.24)$$

where:

$$\bar{V}_{nj} = u_{\infty 0} R \frac{\partial \omega_{sj}}{\partial R} + R \frac{\partial \omega_{sj}}{\partial t} - v_{\infty j}; \quad j = 1, 2.$$

Substitution of Equation (4.24) into Equations (4.11) and (4.12) gives the perturbation forms of the pressure, density, and velocity components on the downstream side

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of the perturbed shock. These resulting expressions must be equated to expressions for the corresponding variables in the field evaluated at the perturbed shock to obtain the proper boundary conditions for the perturbation equations. The resulting conditions are: ($j = 1, 2$ in each equation)

$$p_j \ (R, \ \omega_{so}, \ \varphi, \ t) = \ - \left. \frac{\partial p_o}{\partial \omega} \right|_{\omega = \omega_{so}} \omega_{sj} \ + \ \left. \frac{\partial p_s}{\partial V_{n\infty}} \right|_{V_{n\infty} = -v_{\infty o}} \bar{V}_{nj}$$

$$\rho_j \ (R, \ \omega_{so}, \ \varphi, \ t) = \ - \left. \frac{\partial \rho_o}{\partial \omega} \right|_{\omega = \omega_{so}} \omega_{sj} \ + \ \left. \frac{\partial \rho_s}{\partial V_{n\infty}} \right|_{V_{n\infty} = -v_{\infty o}} \bar{V}_{nj}$$

$$\vec{V}_j \ (R, \ \omega_{so}, \varphi, \ t) = (u_{\infty j} \vec{R} + v_{\infty j} \vec{\omega} + w_{\infty j} \ \vec{\varphi}) - \left(\left. \frac{\partial u_o}{\partial \omega} \right|_{\omega = \omega_{so}} \vec{R} \ + \ \left. \frac{\partial v_o}{\partial \omega} \right|_{\omega = \omega_{so}} \vec{\omega} \right) \omega_{sj}$$

$$+ \Delta V_n \ (-v_{\infty o}, \ p_\alpha, \ \rho_\infty) \left(\vec{R} \ \frac{\partial \omega_{sj}}{\partial R} \ R \ + \ \frac{1}{\sin \omega_{so}} \ \frac{\partial \omega_{sj}}{\partial \varphi} \ \vec{\varphi} \right)$$

$$- \left. \frac{\partial \Delta V_n}{\partial V_{n\infty}} \right|_{V_{n\infty} = -v_{\infty o}} \bar{V}_{nj} \ \vec{r} \tag{4.25}$$

In Equations (4.25), the partial derivatives with respect to ω arise as a result of evaluating the variables in the field at the perturbed shock. The partial derivatives with respect to V_{n1} are to be obtained by differentiation of the expressions indicated in Equation (4.11); expressions for them are given in Reference 9.

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To complete specification of the boundary value problem for the perturbation variables, it is necessary to give the initial conditions at $t = 0$. Corresponding to the assumption stated in Section 4.1, this condition is that the perturbation variables vanish at $t = 0$.

Equations (4.16) and (4.17) together with the boundary conditions at the body surface, Equation (4.18), the boundary conditions at the shock, Equation (4.25), and the initial conditions define two separate, linear boundary value problems for the perturbation variables. In these problems, the "motions" of the body, $\bar{\alpha}(t)$ and $\bar{q}(t)$, appear in inhomogeneous terms and play the role of "driving functions." Because of the linearity of the problems, the solutions for arbitrarily prescribed functions, $\bar{\alpha}$ and $\bar{q}$, can be divided into parts which can be called the particular solution and the complementary solution. The particular solutions are defined as solutions of the inhomogeneous problems which reduce to the trivial solution (all perturbation variables equal zero) when $\bar{\alpha}$ and $\bar{q}$ are identically zero; the complementary solutions are defined as solutions of the homogeneous problems (obtained by setting $\bar{\alpha}$ and $\bar{q}$ equal to zero) which cause the complete solution to satisfy the initial conditions.

On physical grounds, it is known that the complementary solutions must "die out" in time, and their decay time is some nominal multiple of the time it takes the body to move through its own length, L/U_∞. Subsequent to this decay time, the flow field about the body is given by the particular solutions alone. It is, therefore, only the particular solutions which are of interest here.

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The problem of determining the particular solutions of the boundary value problems for arbitrarily prescribed functions, $\overline{\alpha}$ (t) and $\overline{q}$ (t), can be approached in several ways — e.g., by use of the Laplace transform. The approach taken here is one which reduces each problem to a problem of solving an infinite sequence of sets of ordinary differential equations having the ray angle, ω, as an independent variable. The substitutions which accomplish this reduction are tabulated in Appendix I. The form of the pressure perturbation, p_1, is typical:

$$p_1 (R, \omega, \varphi, t) = \sum_{n=o}^{\infty} p_{1,n} (\omega) \left(\frac{R}{U_\infty} \right)^n \frac{d^n \overline{\alpha}}{d t^n} \cos \varphi \qquad (4.26^*)$$

In writing these series, it is assumed that $\overline{\alpha}$ (t) and $\overline{q}$ (t) are analytic functions. Also, it is recognized that there is no a-priori guarantee that these series are convergent for any given functions, $\overline{\alpha}$ (t) and $\overline{q}$ (t). The factor $1/U_\infty^n$ has bee introduced so that the perturbation coefficients, $p_{j,n}$, have the units of pressure.

The two infinite sequences of sets of ordinary differential equations for the variables $p_{j,n}(\omega)$, etc., which arise from the substitutions listed in Appendix I can be written in the common form, (j = 1, 2), (n = 0, 1, 2, 3. . . .).

$$\frac{d}{d\omega} \left\{ \left(\rho_0 \overline{v}_{j,n} + v_0 \rho_{j,n} \right) \sin \omega \right\} + (n + 2 + \delta_{j,2}) \left(\rho_0 u_{j,n} + u_0 \rho_{j,n} \right) \sin \omega$$

$$+ \rho_0 w_{j,n} = \mathscr{R}_{j,n} \qquad (4.27a)$$

*The "zeroth" derivative, $\dfrac{d^o \alpha}{d t^o}$, is equal to $\overline{\alpha}$, by convention.

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$$v_0 \frac{du_{j,n}}{d\omega} - v_0 v_{j,n} + (n + \delta_{j,2}) \left(u_0 u_{j,n} + \frac{p_{j,n}}{\rho_0} \right) = \mathcal{F}_{j,n} \qquad (4.27b)$$

$$v_0 \frac{dv_{j,n}}{d\omega} + u_0 v_{j,n} + v_0 u_{j,n} + v_{j,n} \frac{dv_0}{d\omega} + \frac{1}{\rho_0} \frac{dp_{j,n}}{d\omega} - \frac{\rho_{j,n}}{\rho_0^2} \frac{dp_0}{d\omega} +$$

$$(n + \delta_{j,2}) u_0 v_{j,n} = \mathcal{G}_{j,n} \qquad (4.27c)$$

$$v_0 \frac{dw_{j,n}}{d\omega} + u_0 w_{j,n} + v_0 w_{j,n} \cot \omega - \frac{p_{j,n}}{\rho_0 \sin \omega} + (n + \delta_{j,2}) u_0 w_{j,n} = \mathcal{H}_{j,n} \qquad (4.27d)$$

$$v_0 \frac{dS_{j,n}}{d\omega} + (n + \delta_{j,2}) u_0 S_{j,n} = \mathcal{S}_{j,n} \qquad (4.27e)$$

$$p_{j,n} = c_0^2 \, \rho_{j,n} + e_0^2 \, S_{j,n} \qquad (4.27f)$$

where:

$$\mathcal{R}_{j,n} = -(1 - \delta_{0,n}) U_\infty \, \delta_{j,n-1} \sin \omega$$

$$\mathcal{F}_{j,n} = -(1 - \delta_{0,n}) U_\infty \, u_{j,n-1} - \delta_{1,n} \delta_{j,1} U_\infty^2 \sin \omega + \delta_{0,n} \delta_{j,2} U_\infty (2 v_0 + U_\infty \sin \omega)$$

$$\mathcal{G}_{j,n} = -(1 - \delta_{0,n}) U_\infty \, v_{j,n-1} - \delta_{1,n} U_\infty^2 (\delta_{j,1} \cos \omega + \delta_{j,2}) +$$

$$\delta_{0,n} \delta_{j,2} U_\infty (-2 u_0 + U_\infty \cos \omega)$$

$$\mathcal{H}_{j,n} = -(1 - \delta_{0,n}) U_\infty \, w_{j,n-1} + \delta_{1,n} U_\infty^2 (\delta_{j,1} + \delta_{j,2} \cos \omega) +$$

$$\delta_{0,n} \delta_{j,2} \left\{ 2 U_\infty (u_0 \cos \omega - v_0 \sin \omega) - U_\infty^2 \right\}$$

$$\mathcal{S}_{j,n} = -(1 - \delta_{0,n}) U_\infty \, S_{j,n-1}$$

The condition of irrotationality of the field at zero yaw (Appendix II) has been used to simplify equation (4.27b).

The boundary conditions for these equations can also be written in a common form. At the body, $\omega = \omega_B$, the condition of tangency of the flow becomes:

$$v_{j,n} = 0 \quad \text{at } \omega = \omega_B \tag{4.28}$$

At the shock $\omega = \omega_{so}$, the perturbation variables, $p_{j,n}$ etc., are related to the quantities, $\omega_{j,n}$ which describes the perturbation in the shock shape by:

$$p_{j,n} = \left\{ -\frac{\partial p_o}{\partial \omega}\bigg|_{\omega = \omega_{so}} + (n+1+\delta_{j,2})\, U_\infty \cos \omega_{so} \frac{\partial p_2}{\partial V_{n1}}\bigg|_{V_{n1} = -v_{\infty 0}} \right\} \omega_{j,n}$$
$$+ \lambda_{j,n} \frac{\partial p_2}{\partial V_{n1}}\bigg|_{V_{n1} = -v_{\infty 0}} \tag{4.29a}$$

$$\rho_{j,n} = \left\{ -\frac{\partial \rho_o}{\partial \omega}\bigg|_{\omega = \omega_{so}} + (n+1+\delta_{j,2})\, U_\infty \cos \omega_{so} \frac{\partial \rho_2}{\partial V_{n1}}\bigg|_{V_{n1} = -v_{\infty 0}} \right\} \omega_{j,n}$$
$$+ \lambda_{j,n} \frac{\partial \rho_2}{\partial V_{n1}}\bigg|_{V_{n1} = -v_{\infty 0}} \tag{4.29b}$$

$$S_{j,n} = \frac{p_{j,n}}{e_o^2} - \frac{c_o^2}{e_o^2}\, \rho_{j,n} \tag{4.29c}$$

$$u_{j,n} = (n+1 + \delta_{j,2}) \Delta V_n (-v_{\infty 0}, p_\infty, \rho_\infty) \omega_{j,n} - \delta_{0,n} \delta_{j,1} U_\infty \sin \omega_{so} \qquad (4.29d)$$

$$v_{j,n} = \left\{ -\left(u_0 + \frac{\partial v_0}{\partial \omega} \right) \Big|_{\omega = \omega_{so}} \right.$$

$$- (n+1 + \delta_{j,2}) \ U_\infty \cos \omega_{so} \ \frac{\partial \Delta V_n}{\partial V_{n1}} \Big|_{V_{n1} = -v_{\infty 0}} \Bigg\} \omega_{j,n}$$

$$\left. - \left(\delta_{0,n} U_\infty \left[\delta_{j,1} \cos \omega_{so} + \delta_{j,2} \right] + \lambda_{j,n} \frac{\partial \Delta V_n}{\partial V_{n1}} \Big|_{V_{n1} = -v_{\infty 0}} \right) \right.$$

$$(4.29e)$$

$$w_{j,n} = \left\{ -\frac{\Delta V_n (-v_{\infty 0}, p_\infty, \rho_\infty)}{\sin \omega_{so}} \right\} \omega_{j,n} + \delta_{0,n} U_\infty (\delta_{j,1} + \delta_{j,2} \cos \omega_{so}) \qquad (4.29f)$$

where:

$$\lambda_{j,n} = \delta_{0,n} U_\infty (\delta_{j,1} \cos \omega_{so} + \delta_{j,2}) + (1 - \delta_{0,n}) U_\infty \omega_{j,n-1}$$

Equations (4.27a) through (4.27f) are ordinary differential equations with variable coefficients which are determined by the solution for the axially symmetric conical field, u_0, v_0, p_0 and ρ_0. For $j = 1$ and $n = 0$, the Equations are equivalent to those derived by Stone (Reference 9) for the problem of cones at small yaw. For either value of j and any value of n (except $n = 0$), the equations and boundary conditions are complete provided that the solution for the same value of j and a value of n which is one less, has been previously solved. For $n = 0$, the equations and boundary conditions are complete without a knowledge of the solution for any other value of j and n. For

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any values of j and n, the problem is a two-point boundary value problem in which the boundary conditions contain an unknown constant $\omega_{j,n}$.

The equations have a singular behavior at the body surface where $v_0 = 0$ since v_0 multiplies the derivatives of the perturbation variables in equations (4.27b) through (4.27e). The method of treating this singularity is given in Section 5.

5. NUMERICAL SOLUTION OF THE PROBLEM

The numerical method of solution of the problem is based on manipulated forms of Equations (4.27a) through 4.27f). The objective of these manipulations is to reduce as much as possible of the numerical work to performing quadratures and also to provide a convenient method of handling the singularity at the body surface.

The manipulated forms of the equations are obtained as follows:

Dividing through equation (4.27e) by v_o and solving it by use of an integrating factor gives

$$S_{j,n} = \left(\frac{S_{j,n}}{\psi_o^m}\right)\Bigg|_{\omega = \omega_{so}} \psi_o^m + \psi_o^m \int_{\omega_{so}}^{\omega} \frac{\delta_{j,n}}{v_o \psi_o^m} \, d\omega \qquad (5.1)$$

where

$$m = n + \delta_{2,j}$$

and

$$\psi_o = \sqrt{- \rho_o v_o \sin \omega}$$

has been introduced by use of the equation

$$u_o = -\frac{v_o}{\psi_o} \frac{d\psi_o}{d\omega}$$

from Appendix II.

Multiplying equation (4.27b) by u_o, equation (4.27c) by v_o and equation (4.27e) by T_o, adding the results and combining terms gives

$$v_o \frac{dH_{j,n}}{d\omega} + m u_o H_{j,n} = u_o \mathscr{F}_{j,n} + v_o \mathscr{G}_{j,n} + T_o \mathscr{A}_{j,n} \qquad (5.2)$$

where $H_{j,n}$ is a quantity related to perturbations in the stagnation enthalpy

$$H_{j,n} \equiv \frac{p_{j,n}}{\rho_o} + u_o u_{j,n} + v_o v_{j,n} + T_o S_{j,n} \qquad (5.3)$$

The condition of irrotationality,

$$v_o = \frac{du_o}{d\omega}$$

has been used in deriving equation (5.2). Also a quantity,

$$S_{j,n} \frac{dT_o}{d\omega} - \frac{p_{j,n}}{\rho_o^2} \frac{d\rho_o}{d\omega} + \frac{\rho_{j,n}}{\rho_o^2} \frac{dp_o}{d\omega}$$

which appears in the derivation is shown to be zero by use of equation (4.27f), the relation,

$$\frac{\partial \rho_o}{\partial \omega} = \frac{1}{c_o^2} \frac{\partial \rho_o}{d\omega} \quad ,$$

the identity,

$$\frac{\partial T}{\partial p/s} = \frac{e^2}{\rho^2 c^2}$$

and the fact that

$$S_o = \text{const.}$$

The form of equation (5.2) is the same as equation (4.27e) and the solution is

$$H_{j,n} = \left(\frac{H_{j,n}}{\psi_o^m}\right)\Bigg|_{\omega = \omega_{so}} \psi_o^m + \psi_o^m \int_{\omega_{so}}^{\omega} \frac{u_o \mathscr{F}_{j,n} + v_o \mathscr{G}_{j,n} + T_o \mathscr{I}_{j,n}}{v_o \psi_o^m} \, d\omega \qquad (5.4)$$

Solving eq. (5.4) for $p_{j,n}$, substituting into equation (4.27b) and re-arranging the result gives

$$J_{j,n} \equiv \frac{d \, u_{j,n}}{d\omega} - (m+1) \, v_{j,n} = \frac{\mathscr{F}_{j,n} - m \, H_{j,n} + m \, T_o \, S_{j,n}}{v_o} \qquad (5.5)$$

Substituting for $p_{j,n}$ from equation (5.4) and $v_{j,n}$ from equation (5.5) (in terms of $J_{j,n}$) in equation (4.27d) and re-arranging again gives a result of the same form as equations (4.27e) and (5.2). The solution is

$$W_{j,n} \equiv w_{j,n} + \frac{u_{j,n}}{(m+1) \sin \omega} = \left(\frac{w_{j,n} \sin \omega + \frac{u_{j,n}}{m+1}}{\psi_o^{m+1}}\right)\Bigg|_{\omega = \omega_{so}} \frac{\psi_o^{m+1}}{\sin \omega}$$

$$+ \frac{\psi_o^{m+1}}{\sin \omega} \int_{\omega_{so}}^{\omega} \mathscr{H}_{j,n} \sin \omega + \frac{\left(H_{j,n} - T_o S_{j,n} + \frac{v_o J_{j,n}}{(m+1)}\right)}{m+1} \frac{}{v_o \psi_o} \, d\omega \qquad (5.6)$$

The preceding results can be used to eliminate $v_{j,n}$, $p_{j,n}$ and $W_{j,n}$ from equation (4.27a) and thus reduce it to an equation for $u_{j,n}$. Equation (5.5) gives $v_{j,n}$ in terms of $J_{j,n}$ and $\frac{du_{j,n}}{d\omega}$ and equation (5.4) subsequently gives $p_{j,n}$ in terms of $H_{j,n}$ $u_{j,n}$, $\frac{du_{j,n}}{d\omega}$ and $S_{j,n}$. Substituting this result into equation (4.27f) then gives $p_{j,n}$ in

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terms of the same quantities. Using this result together with the expression for $v_{j,n}$ from equation (5.5) and $w_{j,n}$ in terms of $W_{j,n}$ and $u_{j,n}$ from equation (5.6) in equation (4.27a) gives

$$\frac{d^2 u_{j,n}}{d\omega^2} + \frac{1}{A}\frac{dA}{d\omega}\frac{du_{j,n}}{d\omega} + B u_{j,n} \quad \frac{(m+1)\,\Gamma}{\rho_o\left(1 - \dfrac{V_o^2}{c_o^2}\right)\sin\omega} \tag{5.7}$$

where

$$A = \rho_o\left(1 - \frac{v_o^2}{c_o^2}\right)\sin\omega \;\;\exp\left\{-(2m+3)\int_{\omega_{so}}^{\omega}\frac{u_o v_o}{c_o^2 - v_o^2}\,d\omega\right\} \tag{5.8}$$

$$B = \frac{-1}{\left(1 - \dfrac{v_o^2}{c_o^2}\right)\sin^2\omega} + (m+1)(m+2)\left(\frac{c_o^2 - u_o^2}{c_o^2 - v_o^2}\right) \tag{5.9}$$

$$\qquad - \frac{(m+1)}{\rho_o\left(1 - \dfrac{v_o^2}{c_o^2}\right)\sin\omega}\frac{d}{d\omega}\left(\frac{\rho_o u_o v_o \sin\omega}{c_o^2}\right)$$

$$\Gamma = \frac{d\Gamma_1}{d\omega} + \Gamma_2 \tag{5.10}$$

with:

$$\Gamma_1 = \frac{\rho_o \sin\omega}{(m+1)}\left(1 - \frac{v_o^2}{c_o^2}\right)J_{j,n} - \frac{\rho_o v_o \sin\omega}{c_o^2}H_{j,n}$$

$$\qquad + v_o \sin\omega\left(\frac{\rho_o T_o + e_o^2}{c_o^2}\right)S_{j,n} \tag{5.11}$$

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$$\Gamma_2 = \mathcal{Q}_{j,n} - \rho_0 W_{j,n} - (m+2) \sin \omega \left\{ \frac{\rho_0 u_0}{c_0^2} H_{j,n} \right.$$

$$\left. + \frac{\rho_0 u_0 v_0}{c_0^2} \frac{J_{j,n}}{(m+1)} - u_0 \left(\frac{\rho_0 T_0 + e_0^2}{c_0^2} \right) S_{j,n} \right\}$$

(5.12)

Equations (5.1), (5.4), (5.5) and (5.6) give $S_{j,n}$, $H_{j,n}$, $J_{j,n}$ and $W_{j,n}$ as linear functions of the shock perturbation parameter, $\omega_{j,n}$ by substitution from the boundary conditions, Equations (4.29). That is, they give the variables in the forms:

$$S_{j,n} = S_{j,n}^{(1)} + S_{j,n}^{(2)} \omega_{j,n}$$

$$H_{j,n} = H_{j,n}^{(1)} + H_{j,n}^{(2)} \omega_{j,n}$$

$$J_{j,n} = J_{j,n}^{(1)} + J_{j,n}^{(2)} \omega_{j,n}$$

(5.13)

and:

$$W_{j,n} = W_{j,n}^{(1)} + W_{j,n}^{(2)} \omega_{j,n}$$

where the superscripted variables are known functions of ω (or are known constants in some instances) which can be determined numerically by quadratures. With the super-scripted variables in Equation (5.13) known, the quantities, Γ_1 and Γ_2, which determine the righthand side of Equation (5.7) are also known in the sense that the superscripted quantities in the expressions:

$$\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} \omega_{j,n}$$

(5.14)

$$\Gamma_2 = \Gamma_2^{(1)} + \Gamma_2^{(2)} \omega_{j,n}$$

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are known functions of ω. Thus, Equation (5.7) is a second-order ordinary differential equation of the form:

$$\frac{d^2 u_{j,n}}{d\omega^2} + \frac{1}{A} \frac{dA}{d\omega} \frac{d u_{j,n}}{d\omega} + B u_{j,n} = \tau_1 + \tau_2 \omega_{j,n} \tag{5.15}$$

where τ_1 and τ_2 are known functions of ω, and $\omega_{j,n}$ is an unknown constant. The solution of this Equation involves two additional constants of integration, K_1 and K_2, and can be written in the form

$$u_{j,n} = u_{j,n}^{(1)} + u_{j,n}^{(2)} \omega_{j,n} + g_{j,n} K_1 + z_{j,n} K_2 \tag{5.16}$$

where:

$z_{j,n}$ is a solution of the homogeneous equation,

$$\frac{d^2 z_{j,n}}{d\omega^2} + \frac{1}{A} \frac{dA}{d\omega} \frac{d z_{j,n}}{d\omega} + B z_{j,n} = 0$$

Subject to the conditions,

$$z_{j,n}(\omega_B) = 0, \quad \frac{d z_{j,n}}{d\omega}\bigg|_{\omega = \omega_B} = -2 u_0(\omega_B).$$

(These boundary conditions are chosen so that $z_{1,0} = v_0$);

$g_{j,n}$ is a linearly independent solution of the homogeneous equation (Reference 10, pg 33),

$$g_{j,n} = z_{j,n} \int_{\omega_{so}}^{\omega} \frac{d\omega}{A z_{j,n}^2} \tag{5.17}$$

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and $u_{j,n}^{(1)} + u_{j,n}^{(2)} \, \omega_{j,n}$ is a particular solution of equation (5.15), (Reference 10, page 30)

$$u_{j,n}^{(1)} + u_{j,n}^{(2)} \, \omega_{j,n} = z_{j,n} \int_{\omega_{so}}^{\omega} A g_{j,n} (\tau_1 + \omega_{j,n} \, \tau_2) \, d\omega$$

$$-g_{j,n} \int_{\omega_{so}}^{\omega} A z_{j,n} (\tau_1 + \omega_{j,n} \, \tau_2) \, d\omega$$

The particular solution is separated into the equations

$$u_{j,n}^{(k)} = z_{j,n} \int_{\omega_{so}}^{\omega} A g_{j,n} \, \tau_k \, d\omega - g_{j,n} \int_{\omega_{so}}^{\omega} A z_{j,n} \, \tau_k \, d\omega \qquad (k=1,2)$$

Putting

$$\tau_k = \frac{(m+1)}{\rho_0 \left(1 - \dfrac{v_0^2}{c_0^2}\right) \sin \omega} \left(\frac{d \, \Gamma_1^{(k)}}{d\omega} + \Gamma_2^{(k)}\right)$$

and A from equation (5.8) into the expression for $u_{j,n}^{(k)}$ and integrating by parts to eliminate $\dfrac{d \, \Gamma_1^{(k)}}{d\omega}$ gives

$$u_{j,n}^{(k)} = -(m+1) \, g_{j,n} \left(z_{j,n} \, \Gamma_1^{(k)}\right)_{\omega = \omega_{so}} + (m+1) \left\{ z_{j,n} \, I_g^{(k)} - g_{j,n} \, I_z^{(k)} \right\} \qquad (5.18)$$

where:

$$I_g^{(k)} = \int_{\omega_{so}}^{\omega} \left\{ \Gamma_1^{(k)} \left(\frac{d\, g_{j,n}}{d\omega} + \xi\, g_{j,n} \right) - g_{j,n}\, \Gamma_2^{(k)} \right\} e^{\int_{\omega_{so}}^{\omega} \xi\, d\omega}\; d\omega \qquad (k = 1, 2)$$

$$I_z^{(k)} = \int_{\omega_{so}}^{\omega} \left\{ \Gamma_1^{(k)} \left(\frac{d z_{j,n}}{d\omega} + \xi\, z_{j,n} \right) - z_{j,n}\, \Gamma_2^{(k)} \right\} e^{\overrightarrow{\int_{\omega_{so}}^{\omega} \xi\, d\omega}}\; d\omega \quad (k = 1, 2)$$

$$\xi = -(2m+3)\ \frac{u_o\, v_o}{v_o^2 - c_o^2}$$

The perturbation variables, $v_{j,n}$, can also be expressed in terms of the three constants, $\omega_{j,n}$, K_1 and K_2, by use of Equations (5.5), (5.13), (5.16) and (5.18).

$$v_{j,n} = v_{j,n}^{(1)} + v_{j,n}^{(2)}\ \omega_{j,n} + \frac{K_1}{(m+1)}\ \frac{d g_{j,n}}{d\nu} + \frac{K_2}{(m+1)}\ \frac{d z_{j,n}}{d\omega} \qquad (5.19)$$

where

$$v_{j,n}^{(k)} = -\left(z_{j,n}\, \Gamma_1^{(k)} \right)\Bigg|_{\omega = \omega_{so}} \frac{d g_{j,n}}{d\nu} + \frac{\Gamma_1^{(k)}\, e^{\int_{\omega_{so}}^{\omega} \xi\, d\omega}}{A} - \frac{J_{j,n}^{(k)}}{(m+1)} \qquad (5.19a)$$

$$+\ I_g^{(k)}\ \frac{d z_{j,n}}{d\nu} - I_z^{(k)}\ \frac{d g_{j,n}}{d\nu}$$

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The equations, as written, contain several limiting forms of the type $0 \cdot \infty$ and $\infty - \infty$ as the body surface is approached. With one exception, $J_{j,n}^{(k)}$, these limiting forms have finite values and the final results for the perturbation variable are all finite at the body surface. The following limiting forms are encountered.

1. Equations (5.1), (5.4), and (5.6) contain terms of the type

$$I = \psi_0^m \int_{\omega_{so}}^{\omega} \frac{f(\omega)}{v_0 \psi_0^m} da \quad \text{where } m \neq 0 \text{ (for } m = 0, f \equiv 0 \text{ in all cases) and}$$

$f(\omega)$ is non-singular at the body surface. As the body surface is approached,

$$\psi_0^m \longrightarrow 0 \quad \text{and} \quad \left| \int_{\omega_{so}}^{\omega} \frac{f(\omega)}{v_0 \psi_0^m} d\omega \right| \longrightarrow \infty. \quad \text{However, use of L'Hopitals rule and}$$

the equations of Appendix II gives $\displaystyle \lim_{\imath \to \omega_B} I = \frac{f(\omega_B)}{m \, u_0(\omega_B)}$. These terms are

handled numerically by use of a quadrature formula having the same singularity as the integral.

2. Equation (5.5) requires division by zero at the body surface. For $m = 0$, the numerator of the function is identically zero so the limit is zero; for other values of m, the limit is not finite. However, this infinity does not appear in the subsequent numerical work as explained in items 4 and 5.

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3. Equation (5.17) for $g_{j,n}$ approaches the form $0 \cdot \infty$ at the body surface due to the boundary conditions imposed on $z_{j,n}$. The limit is again finite and is:

$$g_{j,n}(\omega_B) = \frac{1}{2 u_o(\omega_B) A(\omega_B)} \quad .$$

The first derivative of $g_{j,n}$ (which is needed in Equation (5.18) approaches the form $\infty - \infty$ at the body surface and the finite value of this limit is obtained as follows:

differentiating $g_{j,n}$ in equation (5.16) gives

$$\frac{dg_{j,n}}{d\omega} = \frac{1}{Az_{j,n}} + \frac{dz_{j,n}}{d\omega} \int_{\omega_{so}}^{\omega} \frac{d\omega}{Az_{j,n}^2}$$

As an identity we have

$$\frac{1}{Az_{j,n}^2} = -\frac{d}{d\omega}\left\{ \frac{1}{Az_{j,n} \dfrac{dz_{j,n}}{d\omega}} \right\} - \frac{1}{Az_{j,n}\left(\dfrac{dz_{j,n}}{d\omega}\right)^2}$$

$$\left\{ \frac{1}{A}\frac{dA}{d\omega} \frac{dz_{j,n}}{d\omega} - \frac{d^2 z_{j,n}}{d\omega} \right\}$$

or, using the equation for $z_{j,n}$

$$\frac{1}{Az_{j,n}^2} = -\frac{d}{d\omega}\left\{ \frac{1}{Az_{j,n}\dfrac{dz_{j,n}}{d\omega}} \right\} - \frac{B}{A\left(\dfrac{dz_{j,n}}{d\omega}\right)^2}$$

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Substituting this result into the expression for the derivative of $g_{j,n}$

gives

$$\frac{dg_{j,n}}{d_w} = \frac{\dfrac{dz_{j,n}}{d\omega'}}{\left\{Az_{j,n}\dfrac{dz_{j,n}}{d\omega'}\right\}\Bigg|_{\omega' = \omega'_{so}}} + \frac{dz_{j,n}}{d\omega'}\int_{\omega'_{so}}^{\omega'}\frac{B}{A\left(\dfrac{d_{j,n}}{d\omega'}\right)^2}\,d\omega'$$

which is an equivalent expression which does not approach a limiting

form at the body surface.

4. In the computation of $u_{j,n}^{(k)}$, Equation (5.18), integrals of the form:

$$I = \int_{\omega_{so}}^{\omega} \zeta\, J_{j,n}\, d\omega$$

where ζ is free of singularities, are required. Although $J_{j,n}$ is infinite at

the body surface, these integrals are finite. By substituting for $\mathscr{J}_{j,n}$ from

equation (4.27e) in equation (5.2) and solving the result for $\mathscr{A}_{j,n}$, equation

(5.5) can be written:

$$J_{j,n} = \frac{\dfrac{dH_{j,n}}{dr} - T_o\dfrac{dS_{j,n}}{dt} - \mathcal{Y}_{j,n}}{u_o}$$

Using this expression for $J_{j,n}$ and integrating by parts to remove $\dfrac{dH_{j,n}}{d\omega'}$
and $\dfrac{dS_{j,n}}{d\omega'}$ gives the following expression for the integrals

$$I = \left| \frac{\varphi}{u_o} \left(T_o S_{j,n} - H_{j,n} \right) \right|_{\omega = \omega_{so}} + \frac{\varphi}{u_o} \left(H_{j,n} - T_o S_{j,n} \right)$$

$$+ \int_{\omega_{so}}^{\omega} \left\{ S_{j,n} \frac{d}{d\omega} \left(\frac{T_o \varphi}{u_o} \right) - H_{j,n} \frac{d}{d\omega} \left(\varphi / u_o \right) - \frac{\dot{J}_{j,n} \varphi}{u_o} \right\} d\omega$$

There is no problem with singularities with this form for the integrals.

5. At the body surface, the term $\dfrac{\Gamma_1^{(k)} e^{\int_{\omega_{so}}^{\omega} \xi d\omega}}{A} - \dfrac{J_{j,n}^{(k)}}{m+1}$ appearing in

Equation (5.19) assumes the limiting form $\infty - \infty$ (except when $m = 0$). How-

ever, when the expression for $\Gamma_1^{(k)}$ arising from Equation (5.11) is sub-

stituted algebraically into this expression, the infinities cancel without the

application of L'Hopitals rule to give a finite result.

The numerical solution of the problem proceeds in the following order (the axisym-

metric conical flow solution is assumed to have been pre-computed).

1. Compute $S_{1,0}^{(1)}$ and $S_{1,0}^{(2)}$ from the formulae which arise from substituting

the boundary conditions at the shock, Equations (4.29a), (4.29b) and (4.29c)

into Equation (5.1).

2. Compute $H_{1,0}^{(1)}$ and $H_{1,0}^{(2)}$ from the formulae which arise from substituting

the boundary conditions at the shock, Equations (4.29a) through (4.29e) into

Equation (5.4).

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3. Compute $W_{1,0}^{(1)}$ and $W_{1,0}^{(2)}$ from the formulae which arise from substituting the boundary conditions at the shock, Equations (4.29d) and (4.29f), into Equation (5.6).

4. Compute $z_{1,0}$ and $\dfrac{d\,z_{1,0}}{d\,\omega}$ as described following Equation (5.16), using finite difference methods when $m \neq 0$.

5. Compute $g_{1,0}$ and $\dfrac{d\,g_{1,0}}{d\,\omega}$ from Equation (5.17) and its derivative.

6. Compute $I_g^{(1)}$, $I_g^{(2)}$, $I_z^{(1)}$, and $I_z^{(2)}$ and the remaining factors present in Equation (5.18).

7. Compute $v_{1,0}^{(1)}$ and $v_{1,0}^{(2)}$ from Equation (5.19a).

8. Compute the constants, $\omega_{1,0}$, K_1 and K_2 in Equations (5.16) and (5.19) from the boundary conditions, Equations (4.28), (4.29d), and (4.29e).

9. Compute $u_{1,0}$ from Equation (5.16), $v_{1,0}$ from Equation (5.19), $S_{1,0}$, $H_{1,0}$, and $W_{1,0}$ from Equations (5.13), $w_{1,0}$ from Equation (5.6), $p_{1,0}$ from Equation (5.3) and $\rho_{1,0}$ from Equation (4.27f).

10. Repeat the preceding sequence for $n = 1, 2, 3 \ldots N_{max}$ and for $j = 2$, $n = 0,1,2 \ldots N_{max}$ to generate as many perturbation coefficients as desired.

6. SAMPLE RESULTS AND COMPARISONS

Figures 3 through 14* show distributions of the perturbation coefficients defined in the preceding Sections, across the shock layer of a 10-degree cone at Mach 10 for ideal gas conditions. These results show a tendency for the magnitude of the perturbation coefficients to decrease with increasing n. This tendency continues through n = 10 (the highest value for which results have been obtained) and supports the use of series of the type given in Equation (4.26) in obtaining a solution to the unsteady flow problem.

The perturbation coefficients are fairly well behaved. Near the body, some – such as those in entropy, Figures 11 and 12 – have derivatives which vary like $(\omega - \omega_B)^{-1/2}$ due to the v_o which multiplies the derivatives in Equations (4.27b) through (4.27e). This behavior does not cause any difficulty in the numerical work, however.

The perturbation coefficients of the shock angle corresponding to the results shown in Figures 3 through 14 are given in the table below.

TABLE I

PERTURBATION COEFFICIENTS FOR THE SHOCK SHAPE

$$\omega_B = 10^o, \quad M_\infty = 10, \quad \gamma = 1.4$$

$$\omega_{so} = 0.215 \text{ rad}$$

$\omega_{1,0} = -0.0969$	$\omega_{2,0} = -0.176$
$\omega_{1,1} = -0.0742$	$\omega_{2,1} = -0.0112$
$\omega_{1,2} = 0.0319$	$\omega_{2,2} = 0.0104$
$\omega_{1,3} = -0.00737$	$\omega_{2,3} = -0.00250$

---

*All numerical results shown in these and other figures are for an ideal gas with $\gamma = 1.4$.

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If the body were pivoted at 50 percent of its length and oscillated such that

$$\alpha = \bar{\theta} \sin 2\pi ft$$

$$q = 2\pi f \bar{\theta} \cos 2\pi ft$$

with $\dfrac{\pi fL}{U_\infty} \ll 1$, the shock would be given approximately by

$$\omega_s = 0.215 - 0.0969 \left\{ 1 + \left(3.65\frac{x^2}{L^2} - 0.92\frac{x}{L} + 0.125 \right) \left(\frac{2\pi fL}{U_\infty} \right)^2 \right\}$$

$$\sin 2\pi f \left(t + \frac{2.62\,(x-0.191L)}{U_\infty} \right) .$$

The amplitude of oscillation in the shock is essentially the quasi-steady value associated with the cone at small yaw under steady-state conditions and the time lag, $\dfrac{2.62(x-0.191L)}{U_\infty}$ is quite small, in general.

Figures 15 through 28 give static and dynamic force and moment derivatives resulting from integration of surface pressures. The static normal force derivatives given in figure 15 are identical to those given in References 11 and 12. The normal force derivative $C_{N_{qo}}$, applies for rotation about the cone vertex. For other locations of the center of gravity it is given by:

$$C_{N_q} = C_{N_{qo}} - 2\frac{x_{cg}}{L} C_{N_\alpha}$$

Figures 29 and 30 show comparisons of the numerical results for unsteady normal force derivatives with results obtained by several other methods. These are: 1) the

second-order potential theory due to Tobak and Wehrend (Reference 13); 2) the unsteady flow theory due to Zartarian, Hsu and Ashley (Reference 14); and, 3) the Newtonian impact theory. In order to obtain the form of results shown in Figures 29 and 30 from the results of Zartarian, et. al., it was necessary to substitute expansions of the form

$$\alpha \left(t - {}^x/U_\infty\right) = \alpha(t) - \dot{\alpha}(t) \frac{x}{U_\infty} + \dots$$

in Equation (38) of Reference 14.

Figure 29 shows a fair agreement between the present results and potential theory for a 10-degree cone at low Mach numbers. The agreement for a 20-degree cone (Figure 30) is not so good, however. At higher Mach numbers, the impact theory does not predict $C_{N_{qo}}$ and $C_{N_{\dot{\alpha}}}$ as well as might be expected based on results for C_{N_α}. The agreement with the theory of Reference 14 is good over limited Mach No. ranges for both the 10-degree and 20-degree cones but the two sets of results tend to diverge at both high and low Mach numbers.

Figure 31 shows a comparison of numerical results with experimental data for a 10-degree cone at Mach 10 over a range of center of gravity locations. The data also cover a range of Reynolds numbers. The agreement between theory and experiment is quite good except for the highest Reynolds number. Figure 32 shows the experimentally determined variation of the dynamic stability parameter, $C_{m_q} + C_{m_{\dot{\alpha}}}$, with Reynolds number and gives some indication that the disagreement just noted may be due to boundary layer transition.

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Figure 33 shows a comparison of theoretical results and experimental data for a 20-degree cone at Mach 8. The data are for varying angle of attack but show very little influence of angle of attack. The agreement between theory and experiment at zero angle of attack is seen to be quite good.

Figure 34 shows the theoretically predicted effect of frequency of oscillation on the dynamic stability parameter, $C_{m_q} + C_{m_{\dot{\alpha}}}$, for a 10-degree cone in a forced-oscillation wind tunnel experiment. The values of the constants, K_1, K_2, etc., indicated on the Figure 34 are related to the unsteady force coefficients, $C_{N_{\ddot{\alpha}}}$ $C_{N_{\dot{q}}}$, etc. The result is that there is no detectable effect of frequency over the range of frequencies encountered in wind tunnel testing. This is not in agreement with experimental observations which show a fairly large effect of frequency (Reference 15). The effects of frequency appear to be a strong function of Reynolds number.
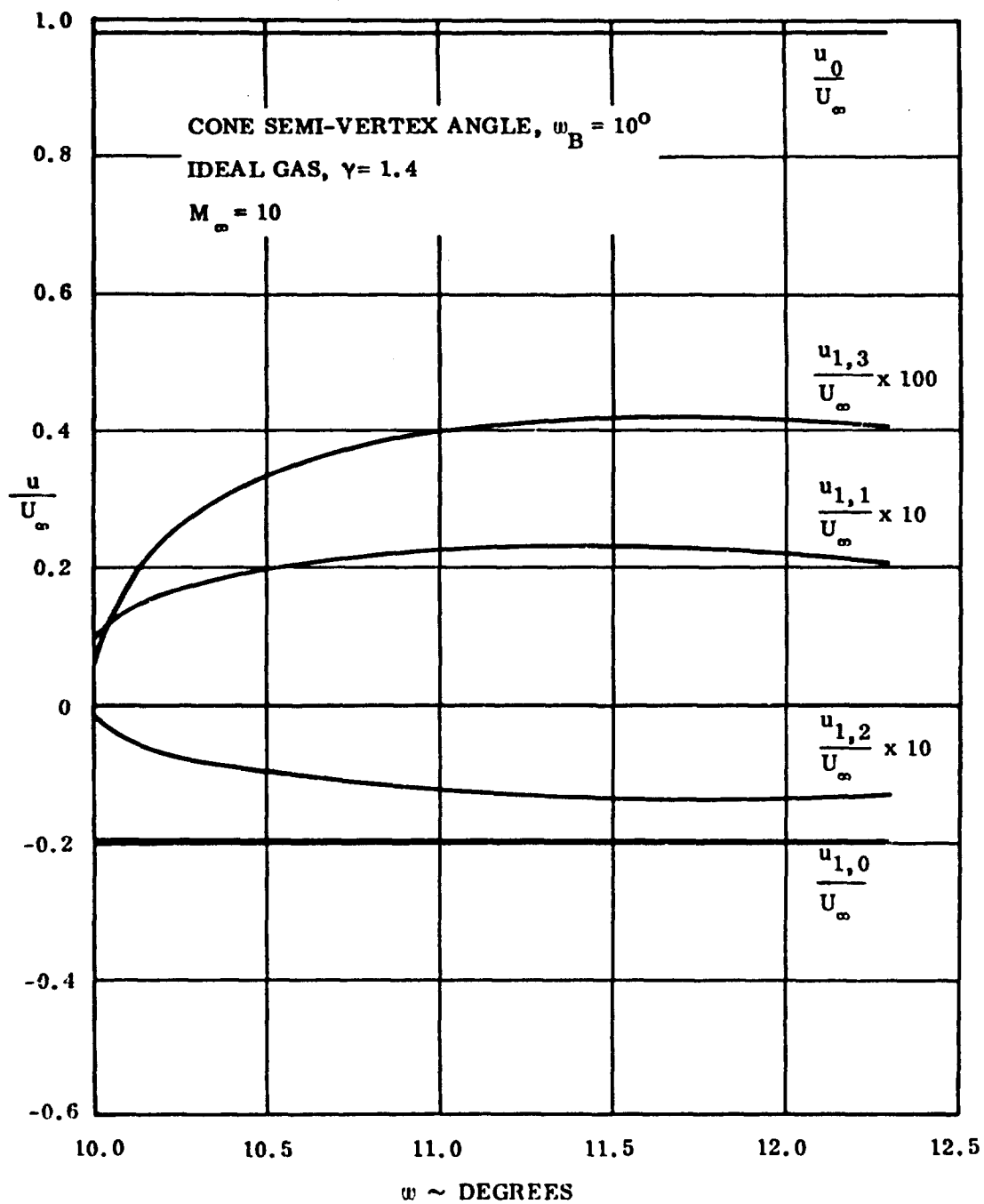
Figure 3. Perturbations in the Velocity Component, u
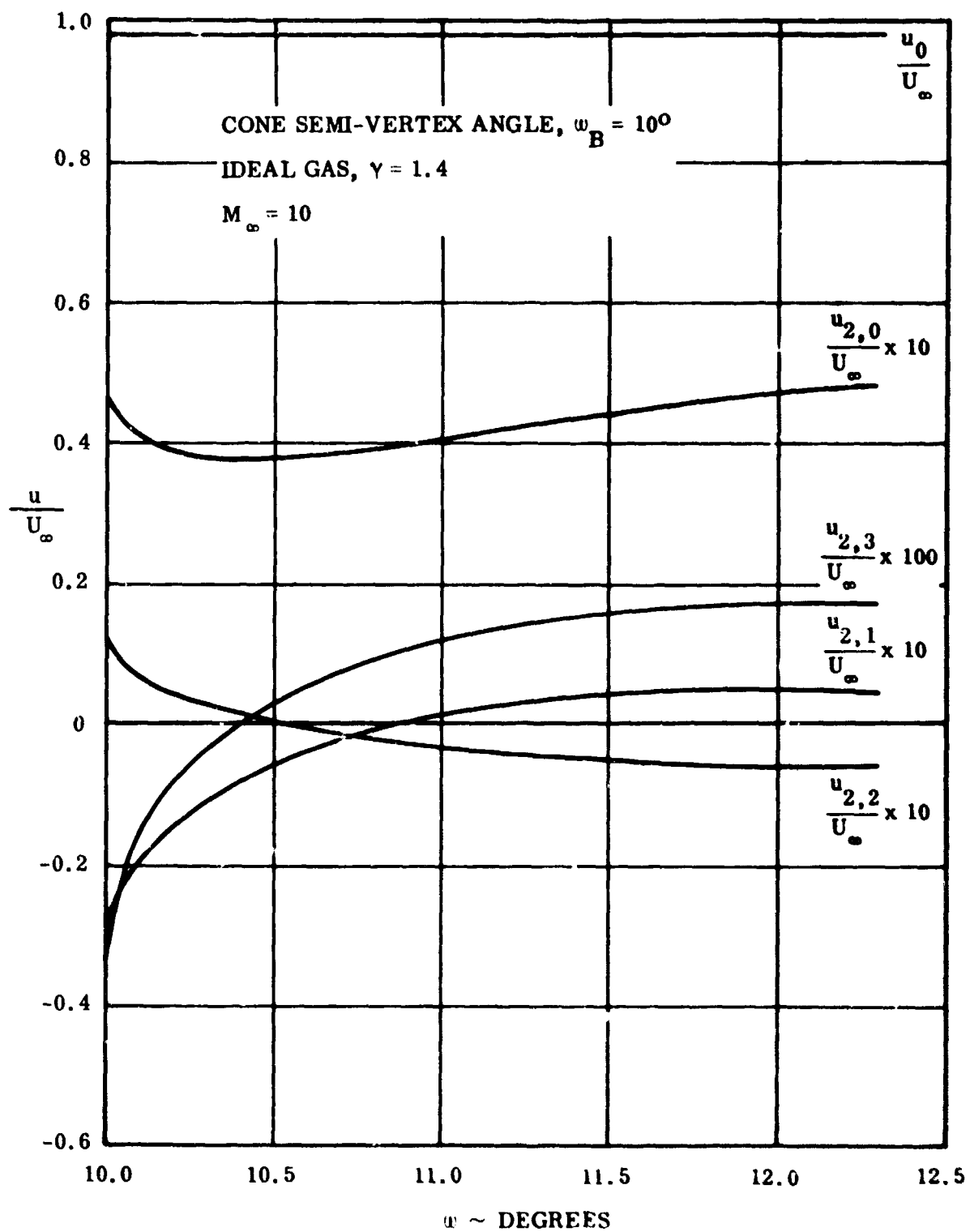
Figure 4. Perturbations in the Velocity Component, u

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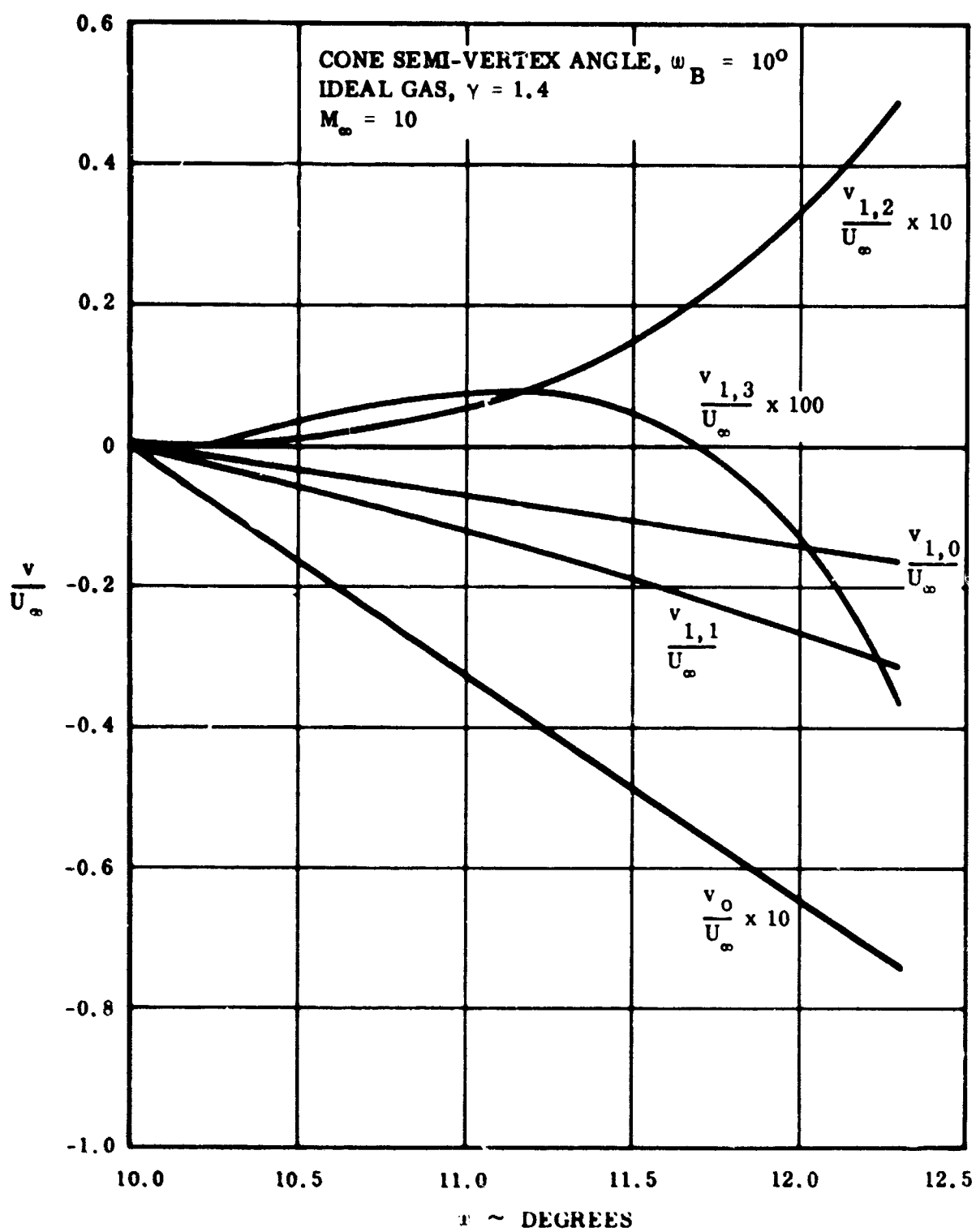
Figure 5.   Perturbations in the Velocity Component, v

42
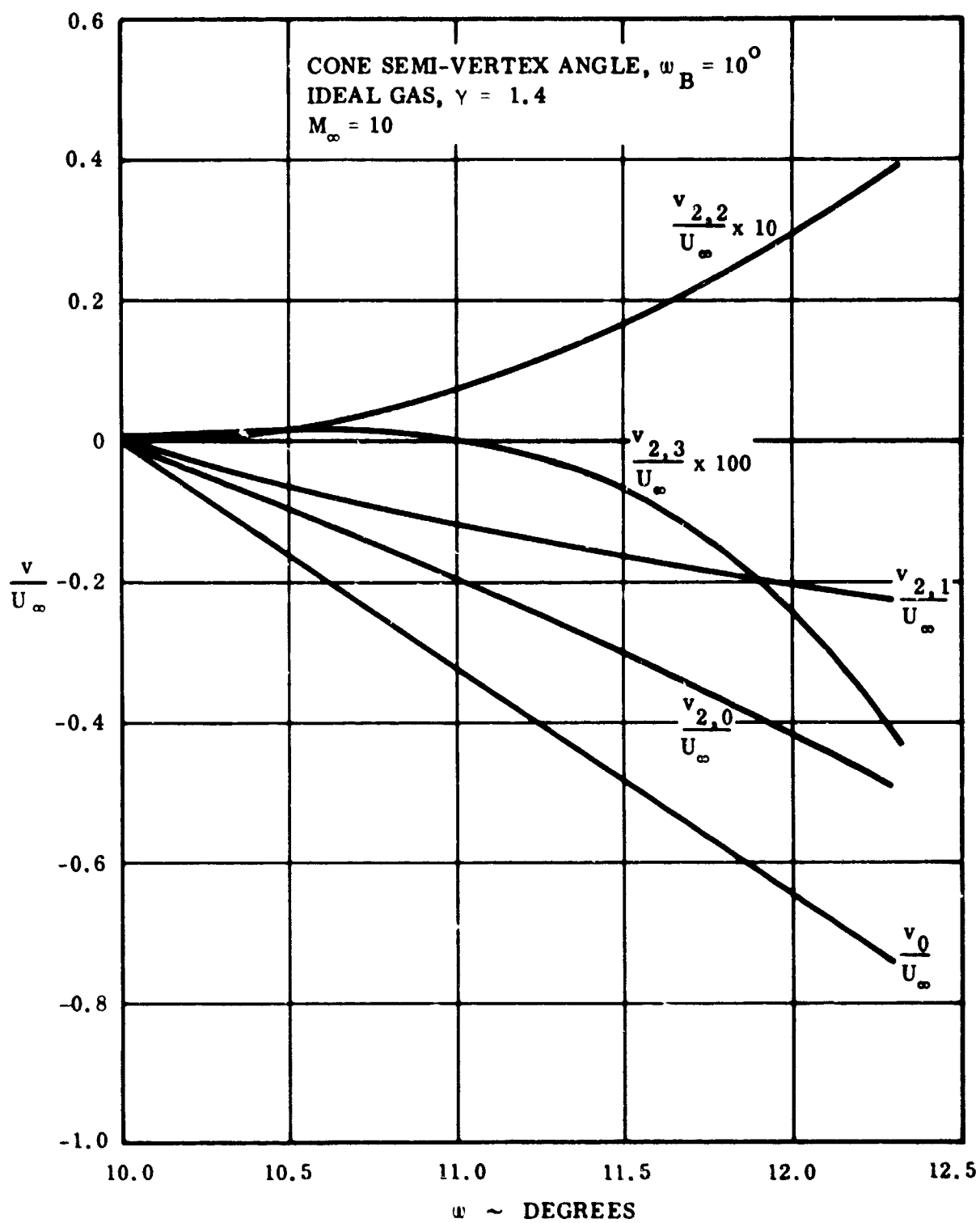
Figure 6. Perturbations in the Velocity Component, v

43
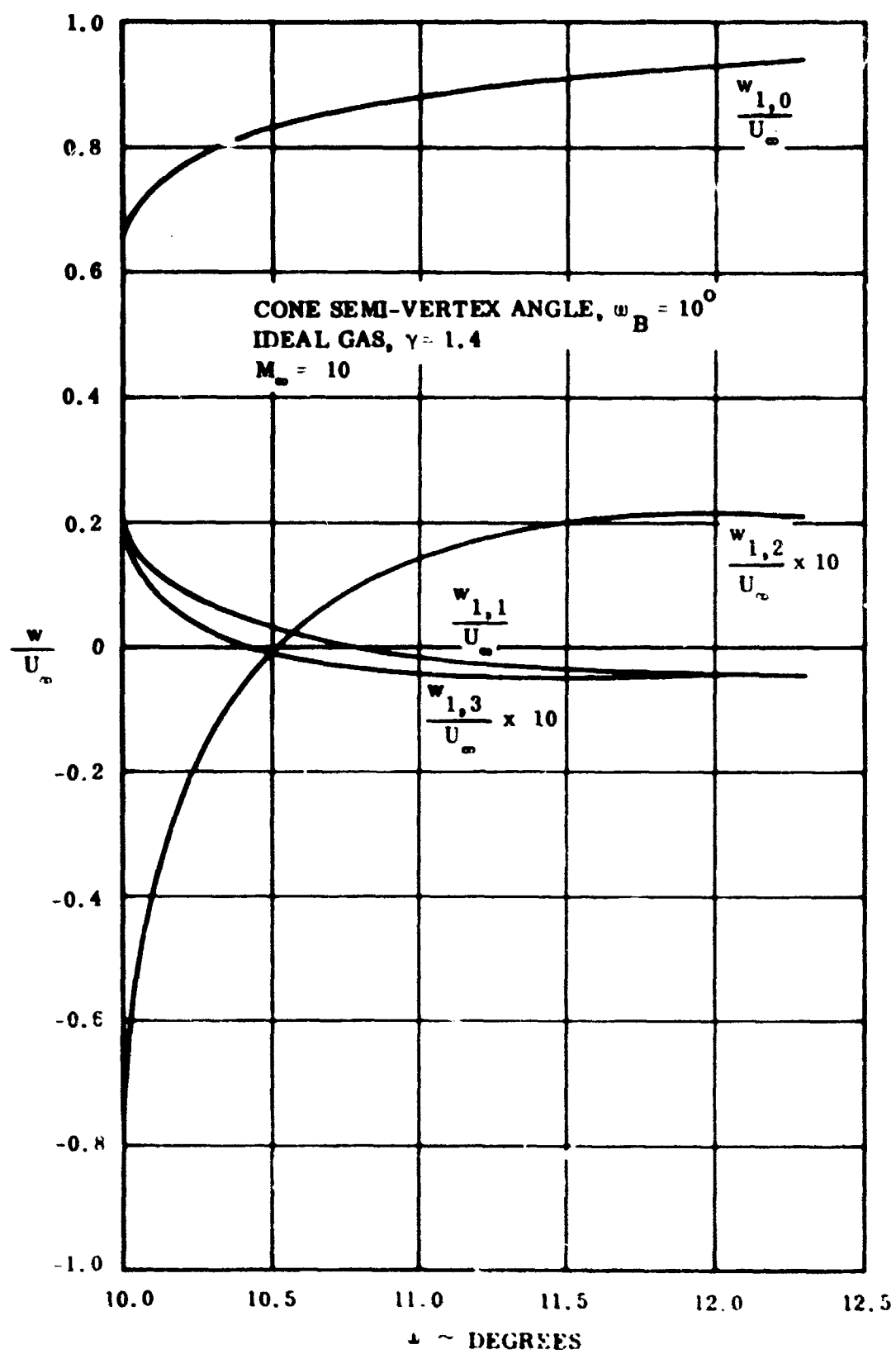
Figure 7.   Perturbations in the Velocity Component, w

44
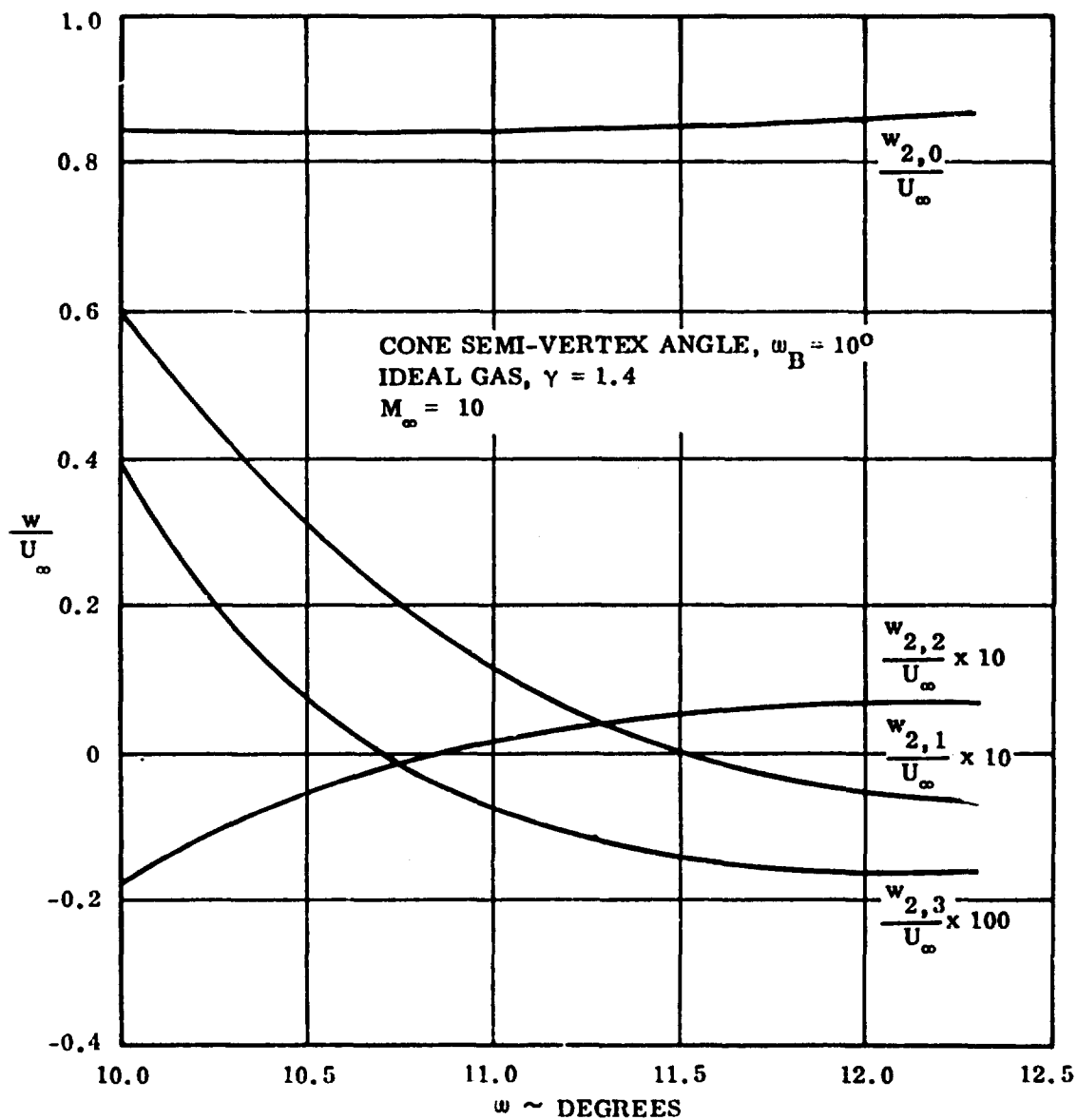
Figure 8.   Perturbations in the Velocity Component, w

45
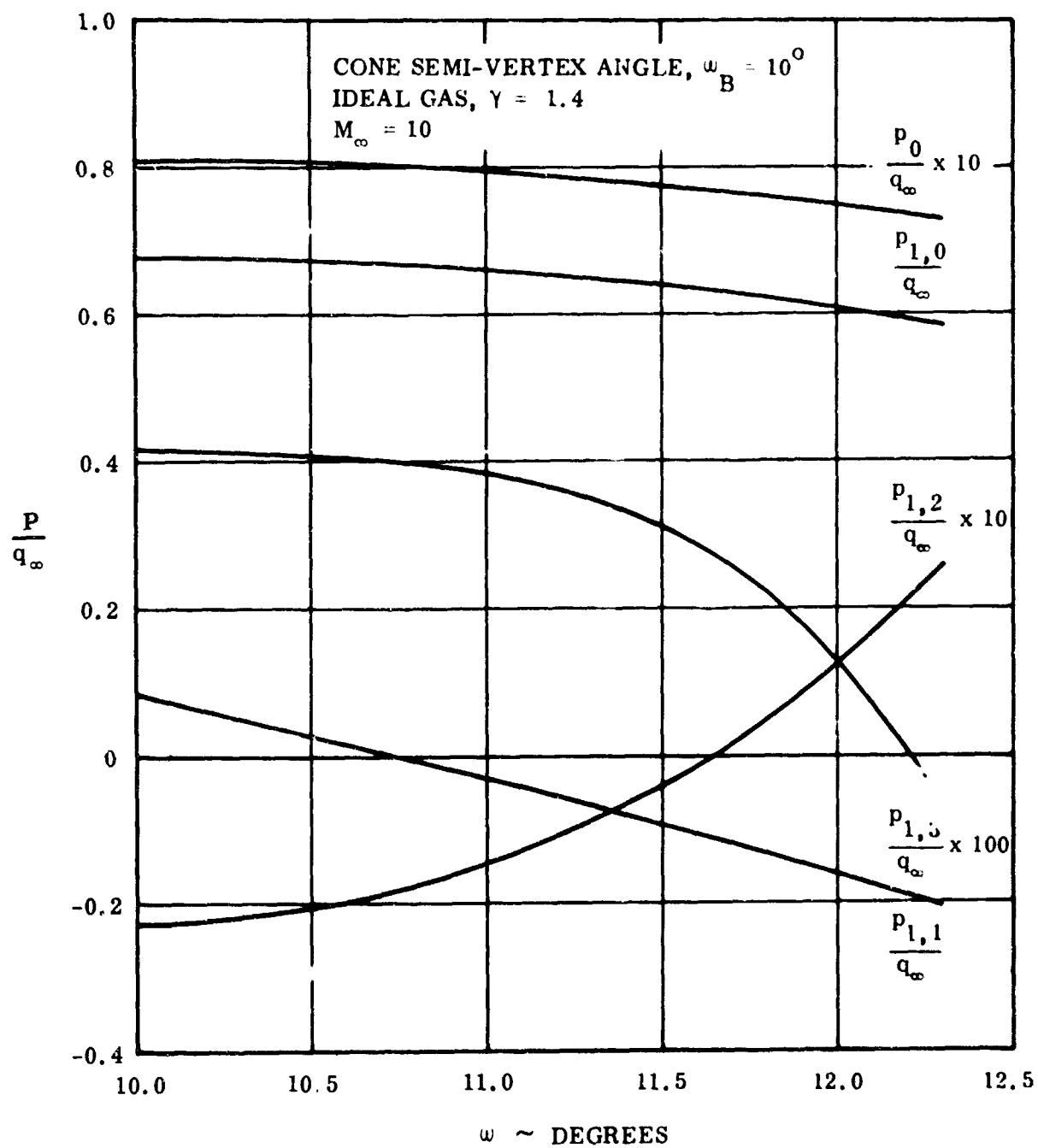
Figure 9. Perturbations in the Pressure

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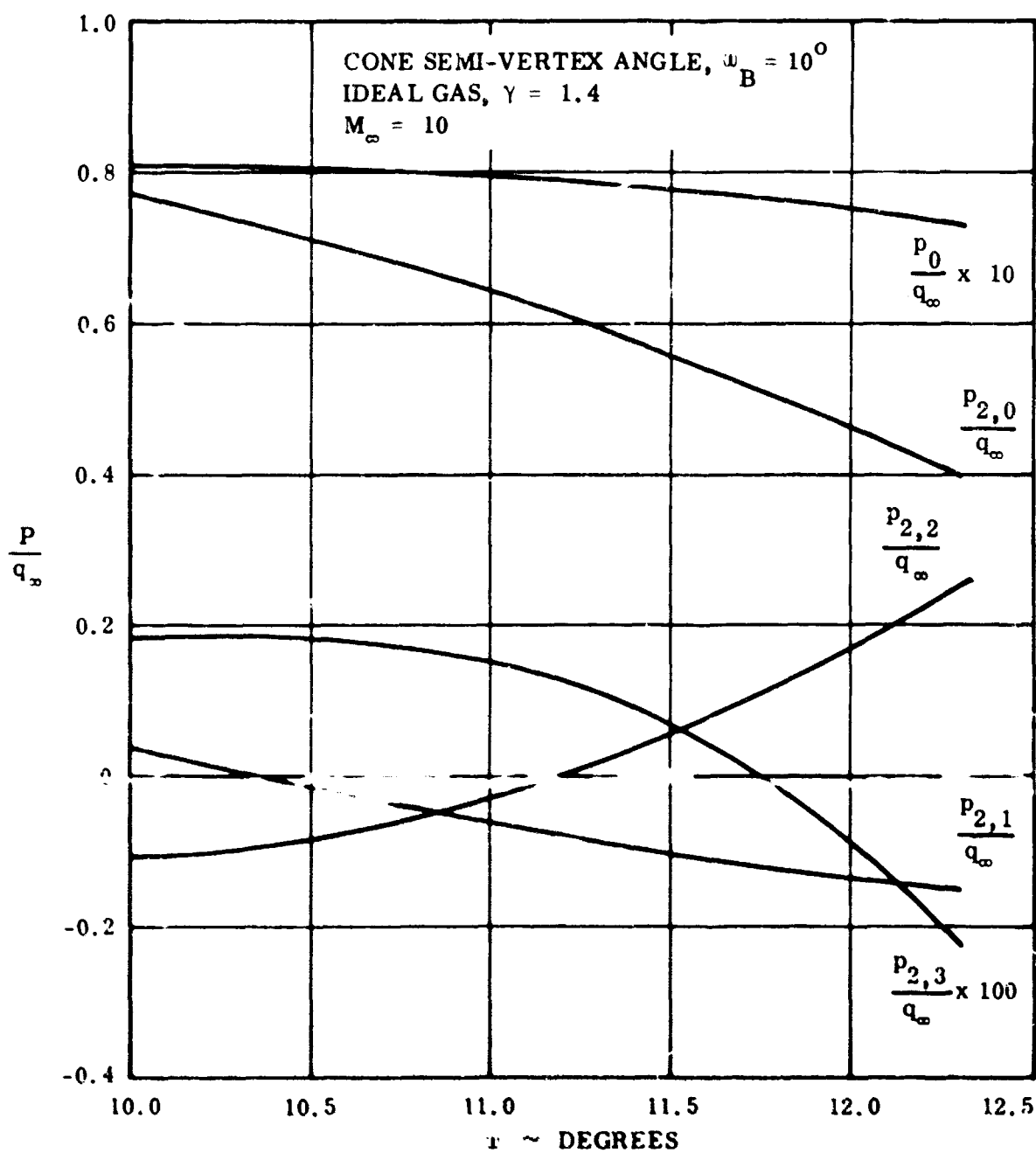
Figure 10. Perturbations in the Pressure

CONE SEMI-VERTEX ANGLE, $\omega_B = 10^O$
IDEAL GAS, $\gamma = 1.4$
$M_\infty = 10$

$S/R_{1,0}$

$S/R_0 \times 10$

$S/R_{1,2}$

$S/R_{1,3}$

$S/R_{1,1}$

$\frac{S}{R}$

$\omega \sim$ DEGREES
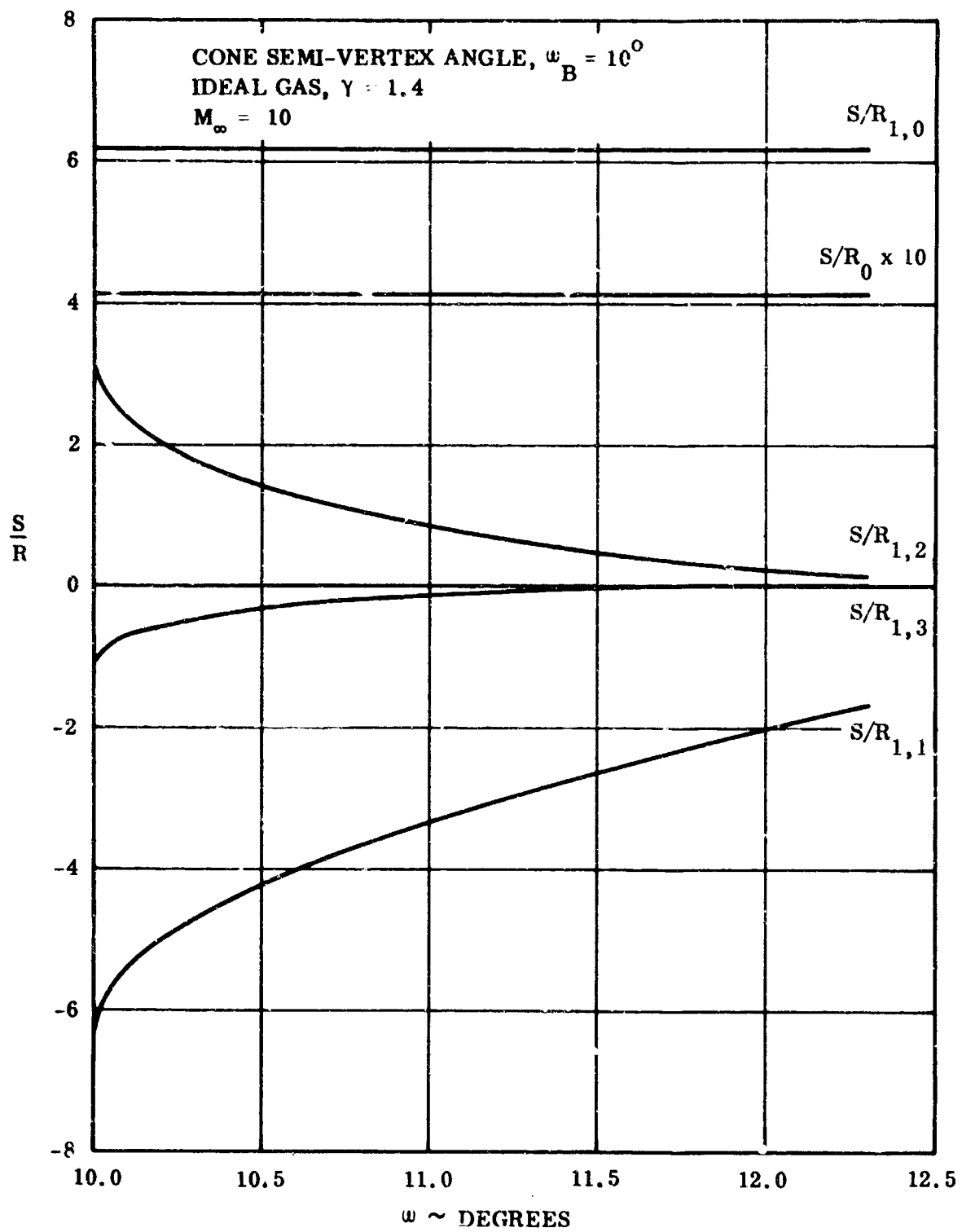
Figure 11. Perturbations in the Entropy

48
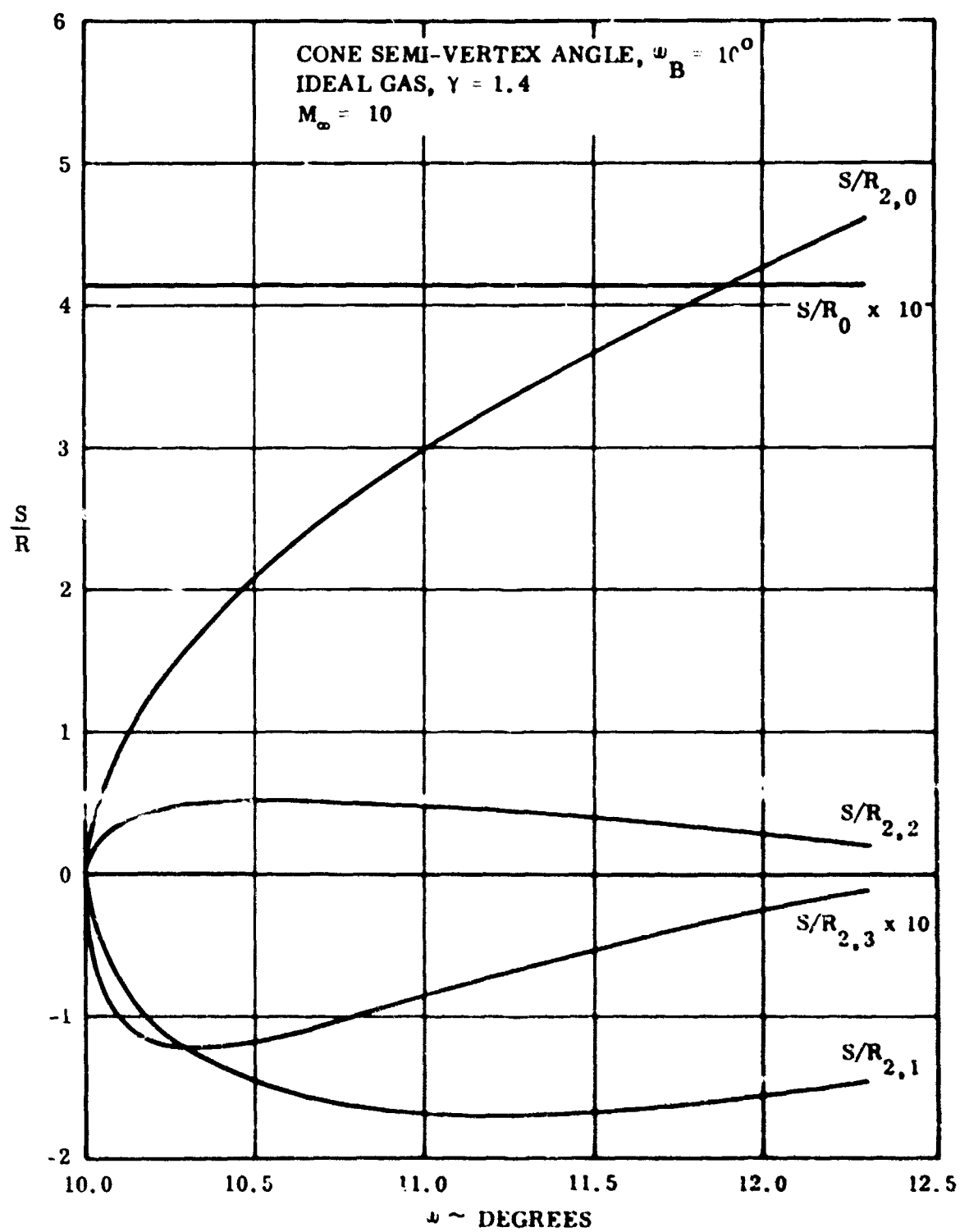
Figure 12. Perturbations in the Entropy
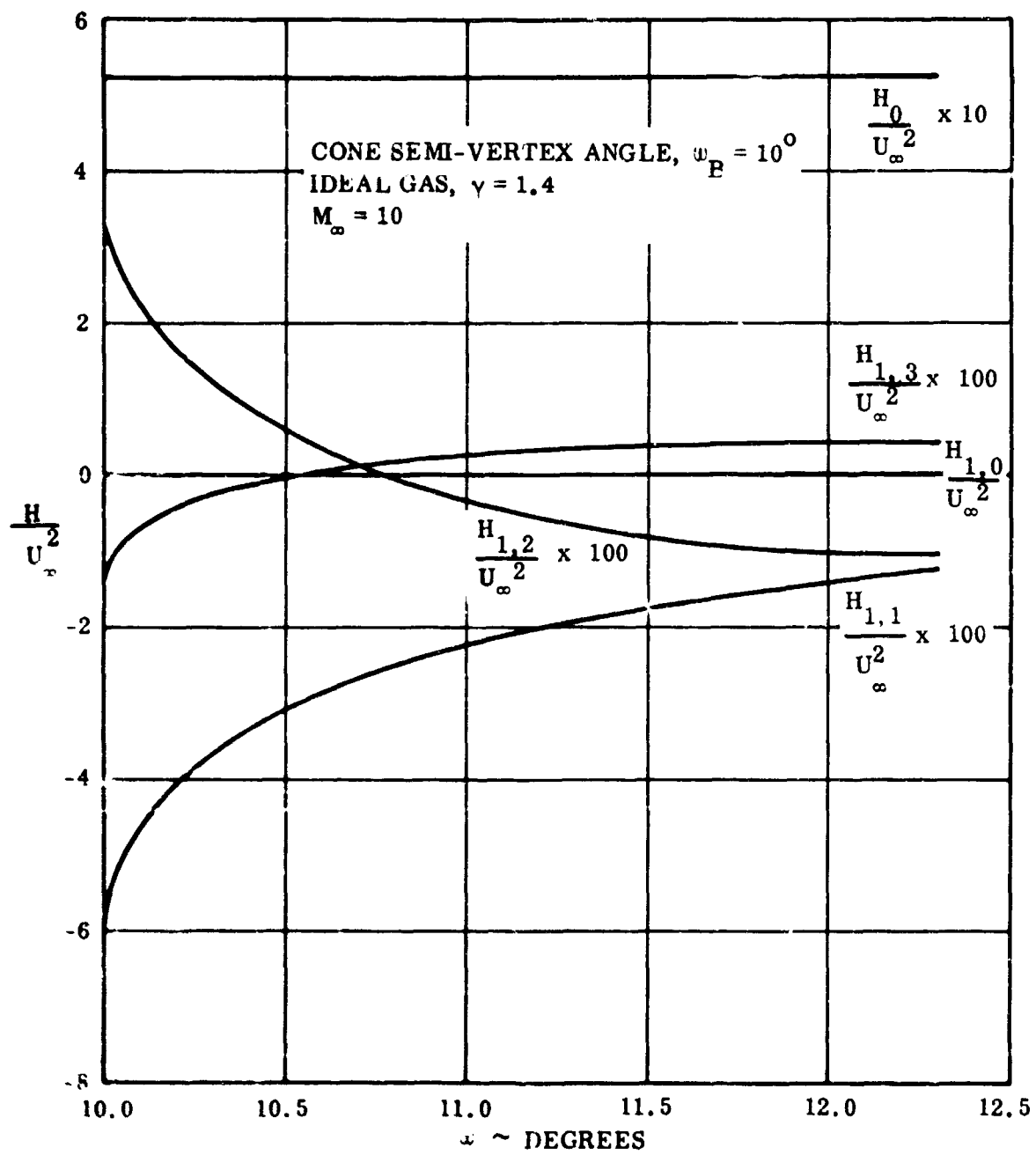
Figure 13. Perturbations in the Stagnation Enthalpy

50
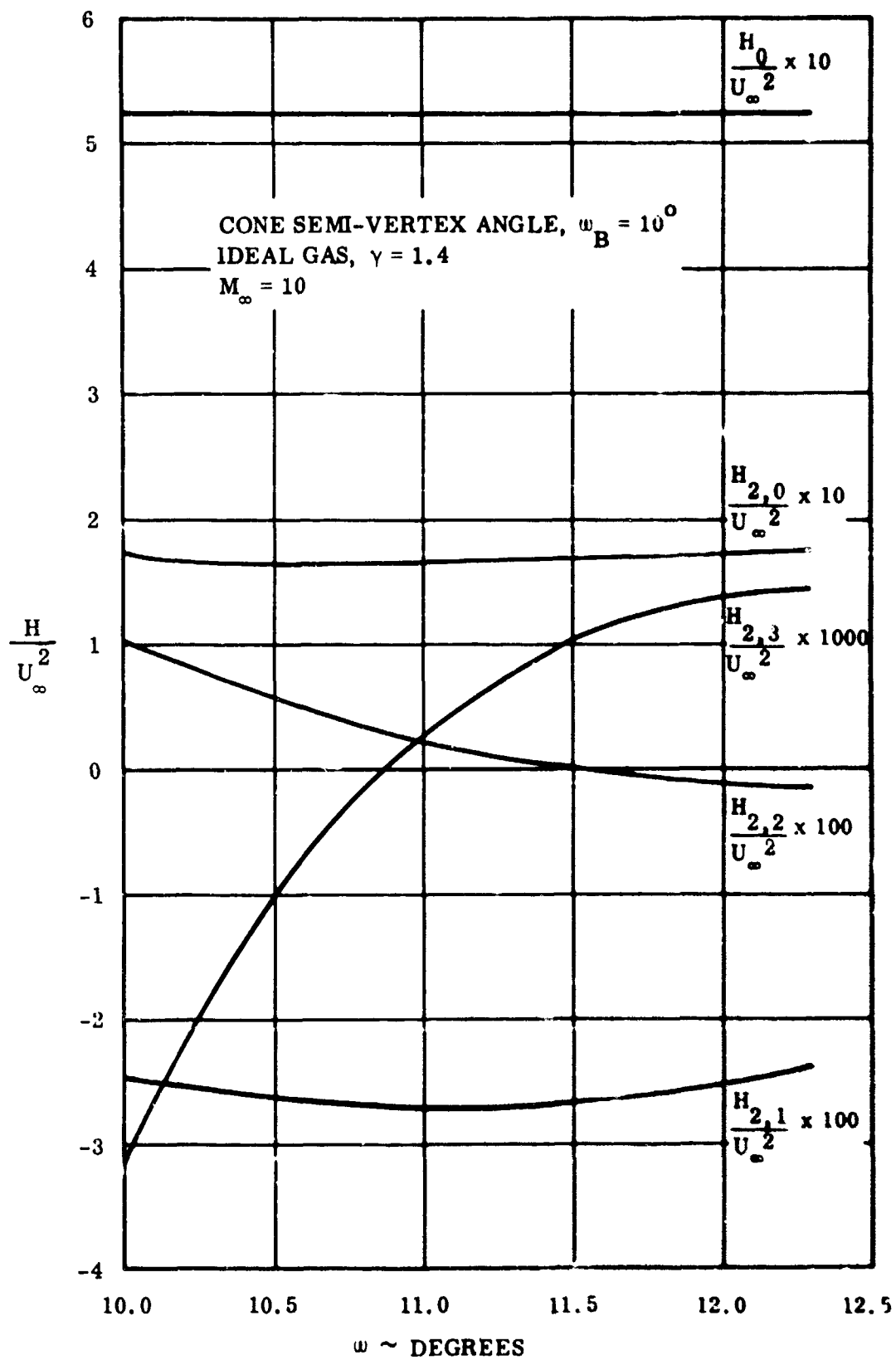
Figure 14. Perturbations in the Stagnation Enthalpy
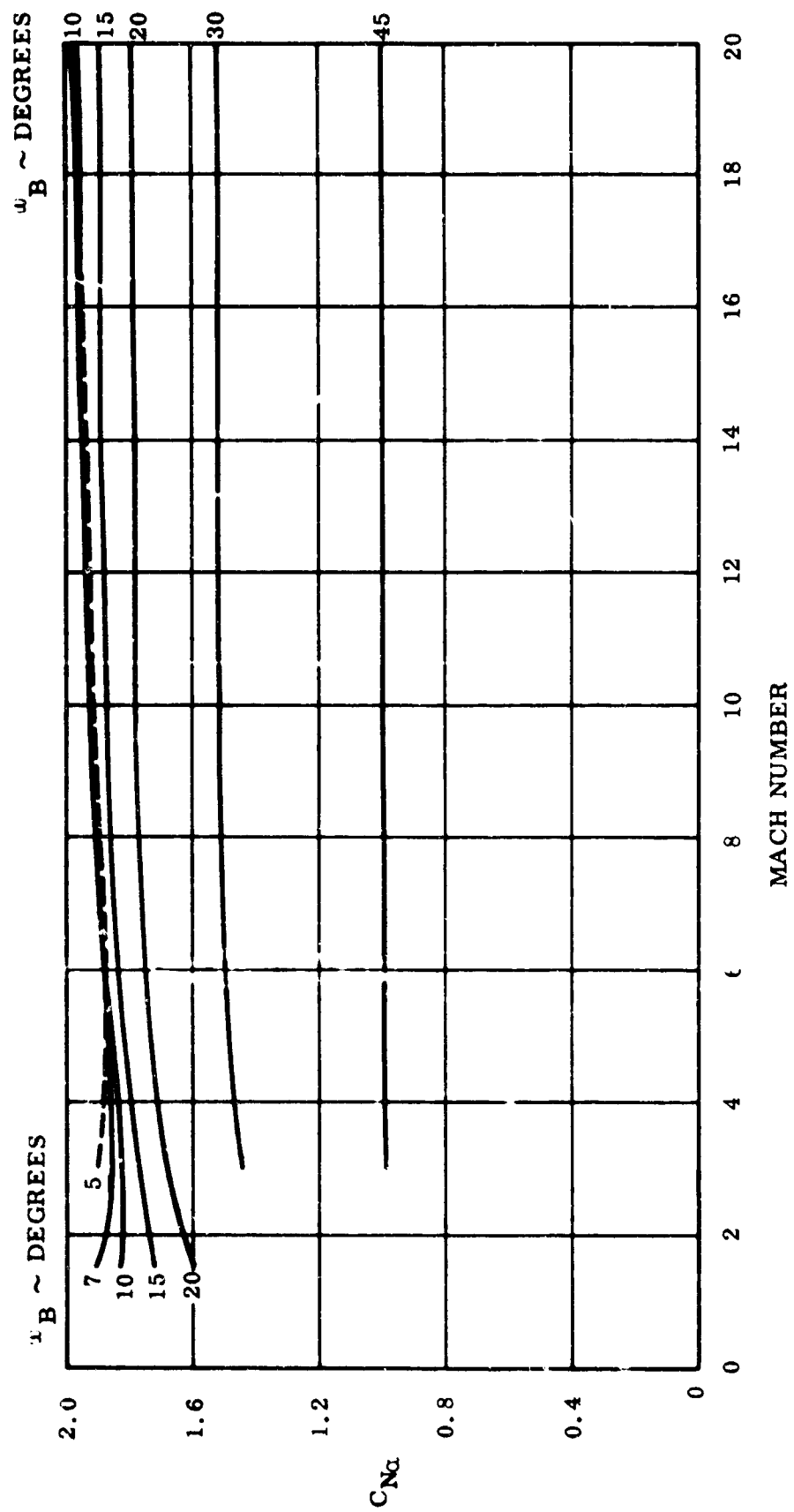
Figure 15. Static Normal Force Derivatives, C_{N_α}

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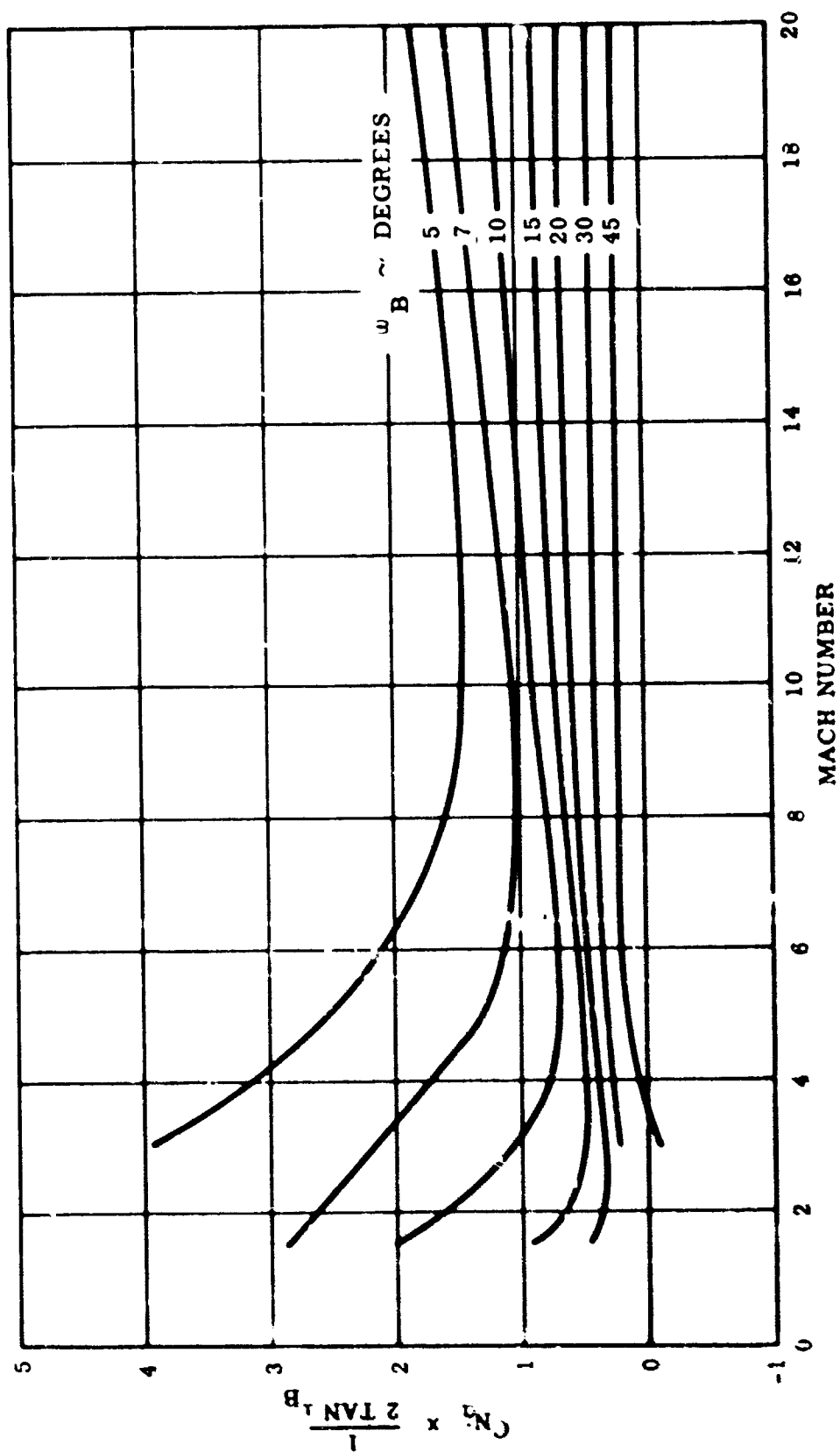
Figure 16. Dynamic Normal Force Derivatives, C_{N_α}
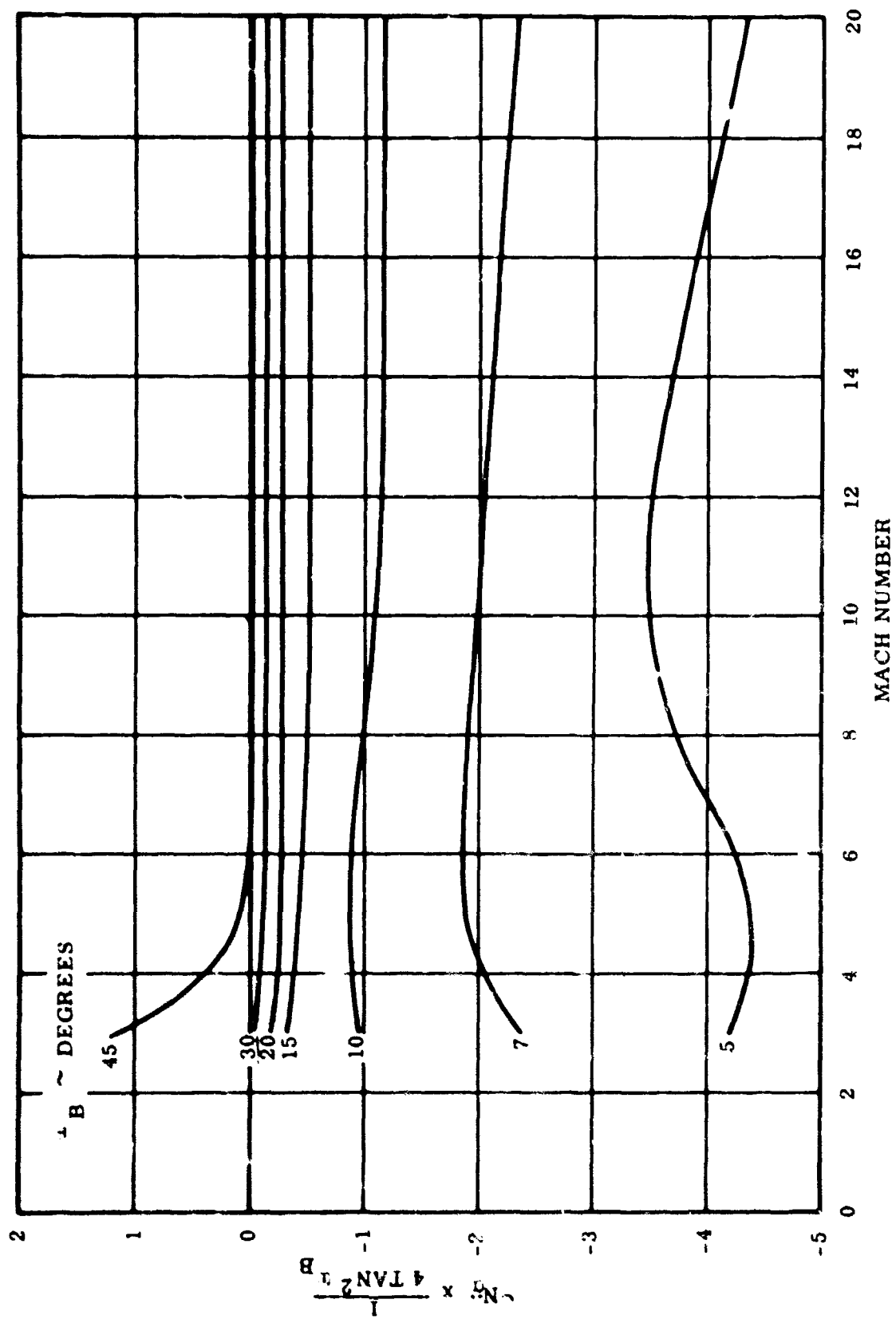
Figure 17. Dynamic Normal Force Derivatives, $C_{N_{\dot{\alpha}}}$
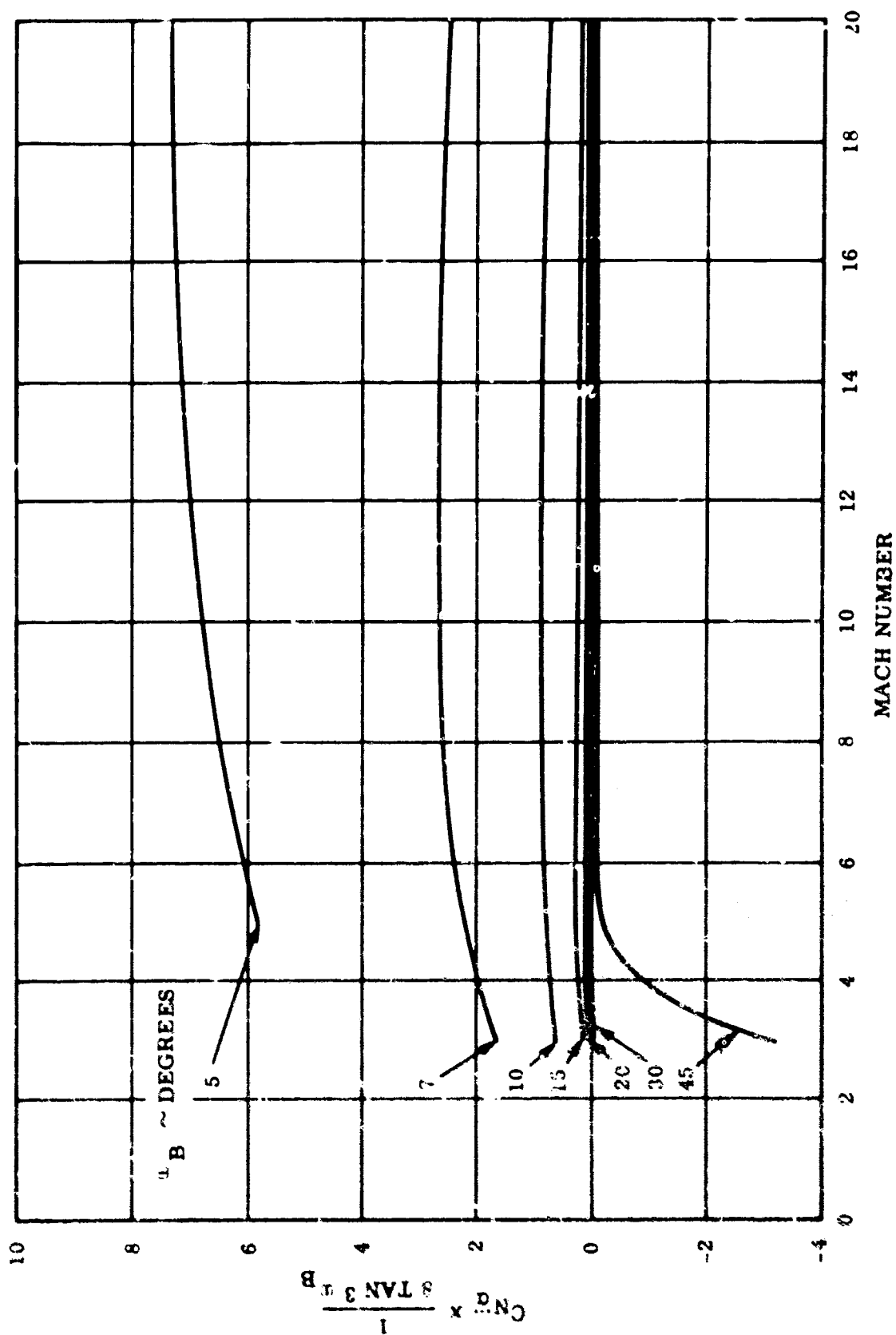
Figure 18. Dynamic Normal Force Derivatives, $C_{N_{\dot{\alpha}}}$.

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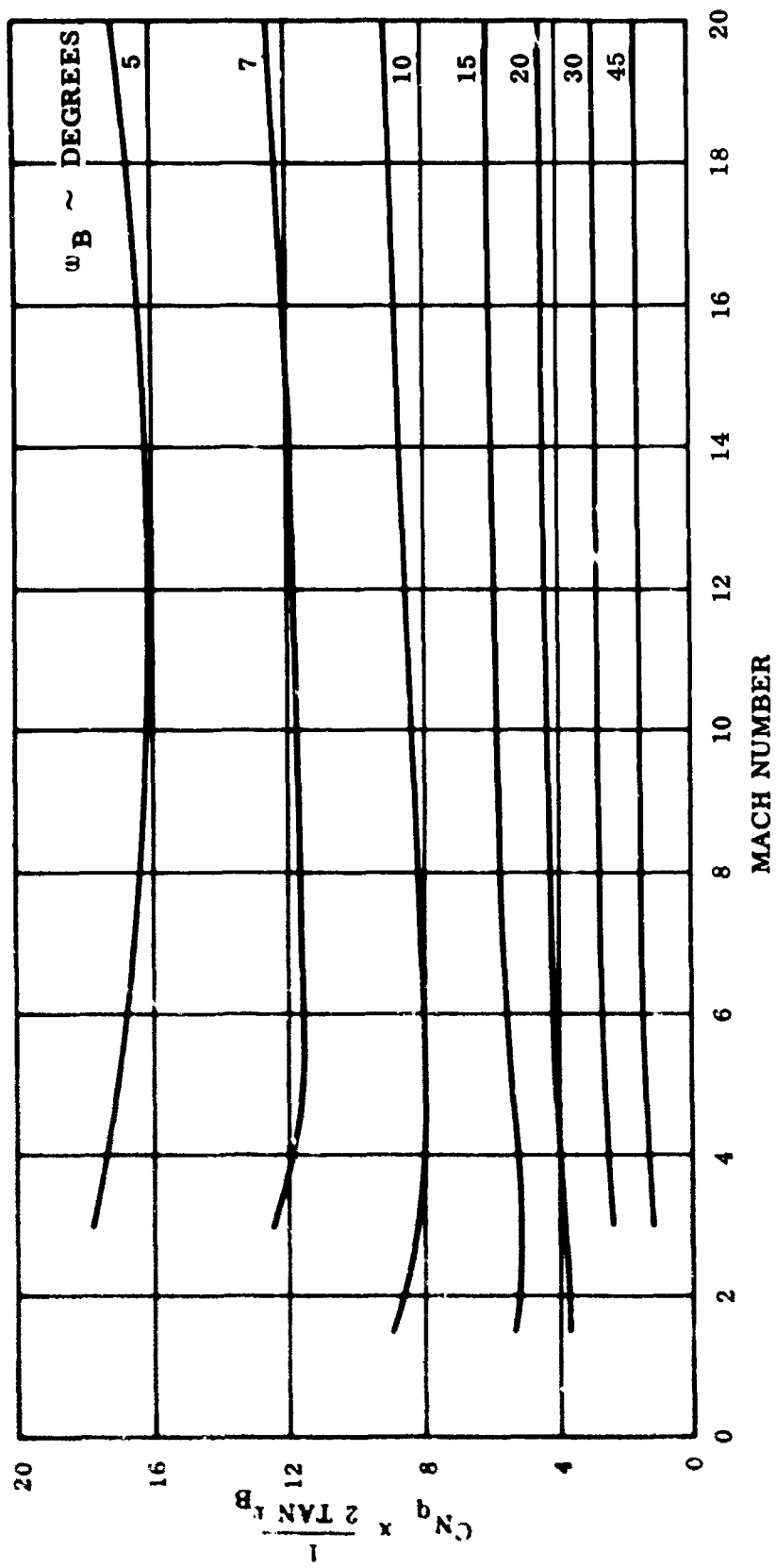
Figure 19. Dynamic Normal Force Derivatives, C_{N_q}
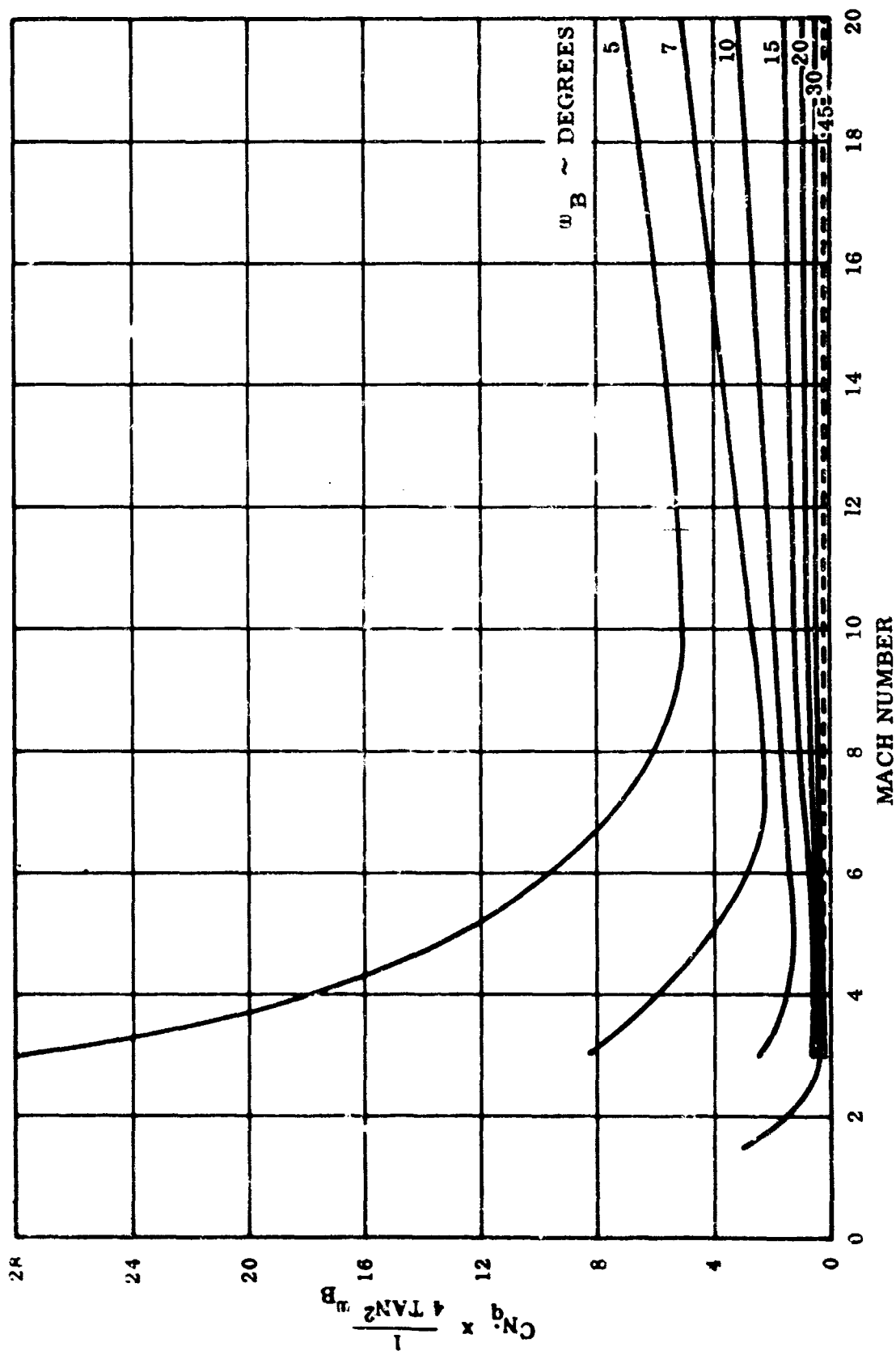
Figure 20. Dynamic Normal Force Derivatives, C_{N_q}
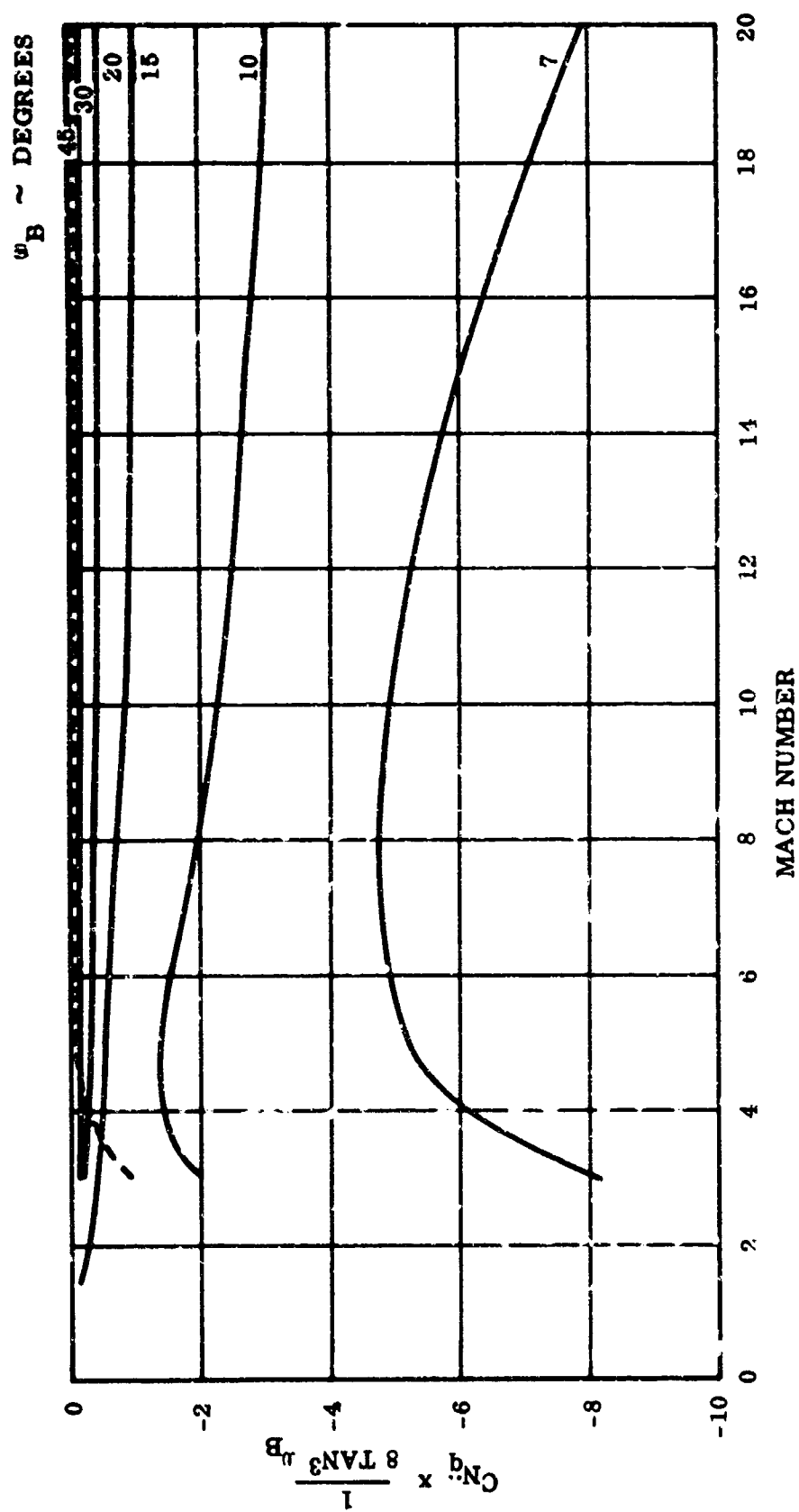
Figure 21. Dynamic Normal Force Derivatives, $C_{N_{\dot{q}}}$
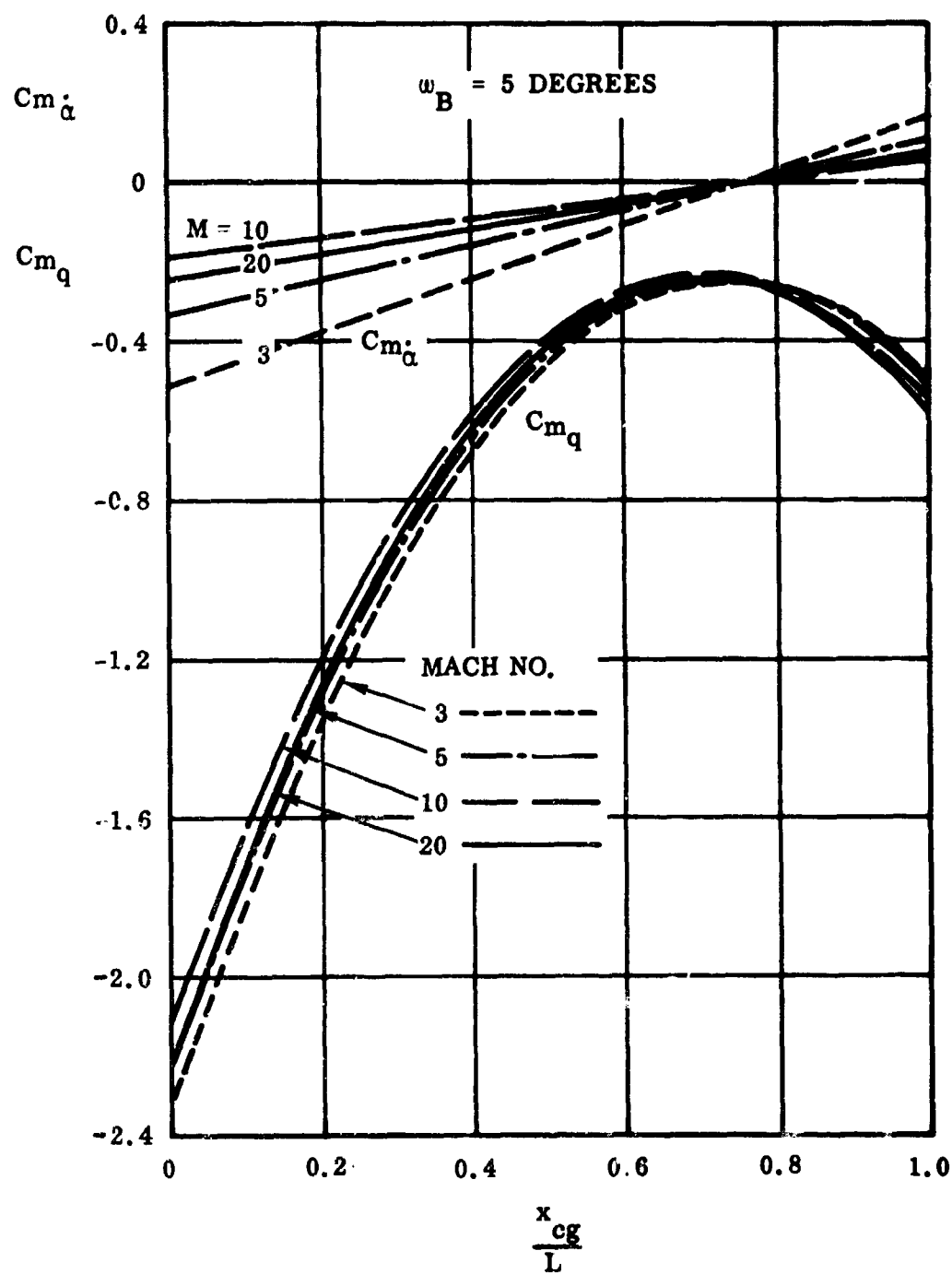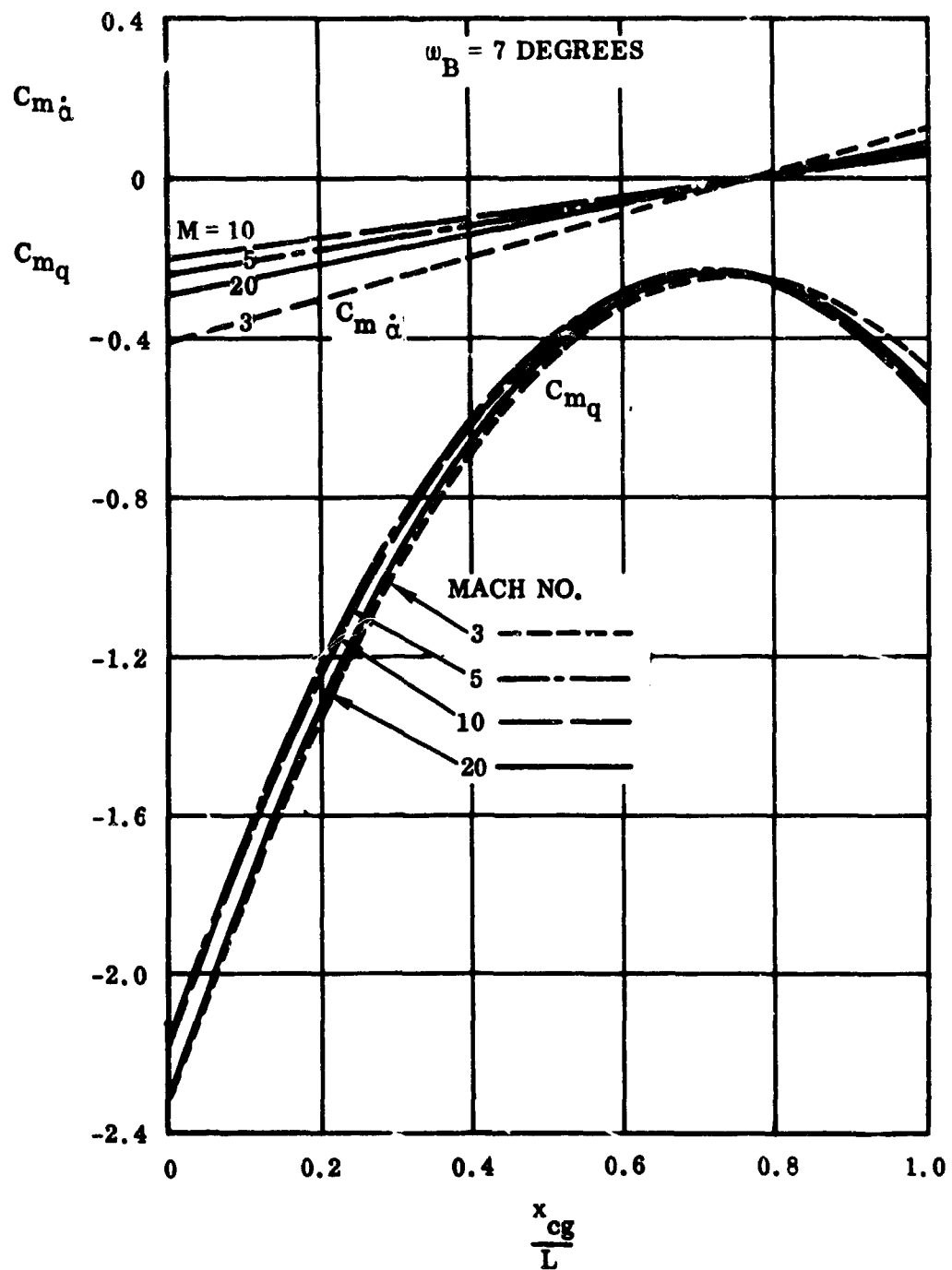
Figure 22. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$

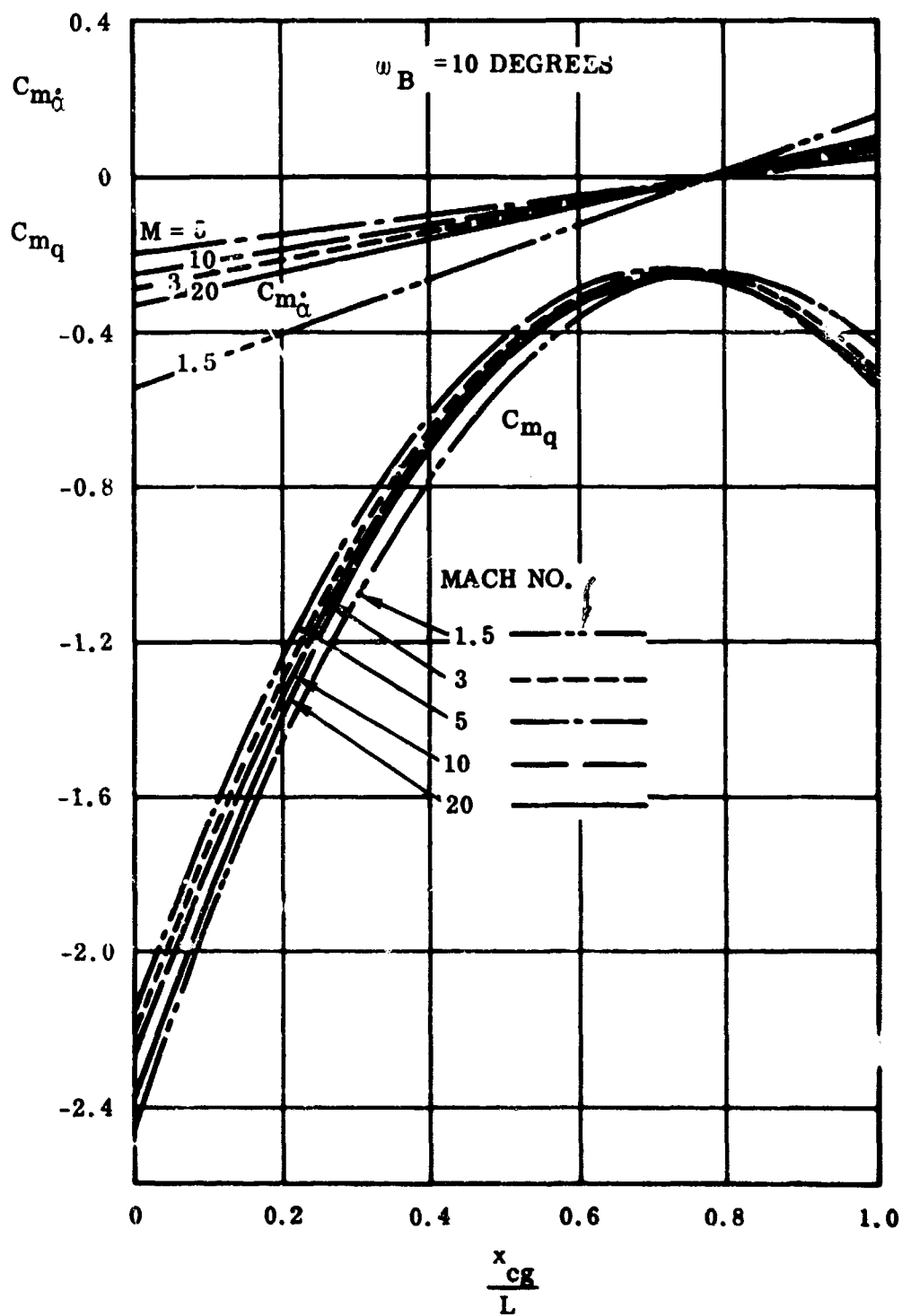Figure 23. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$

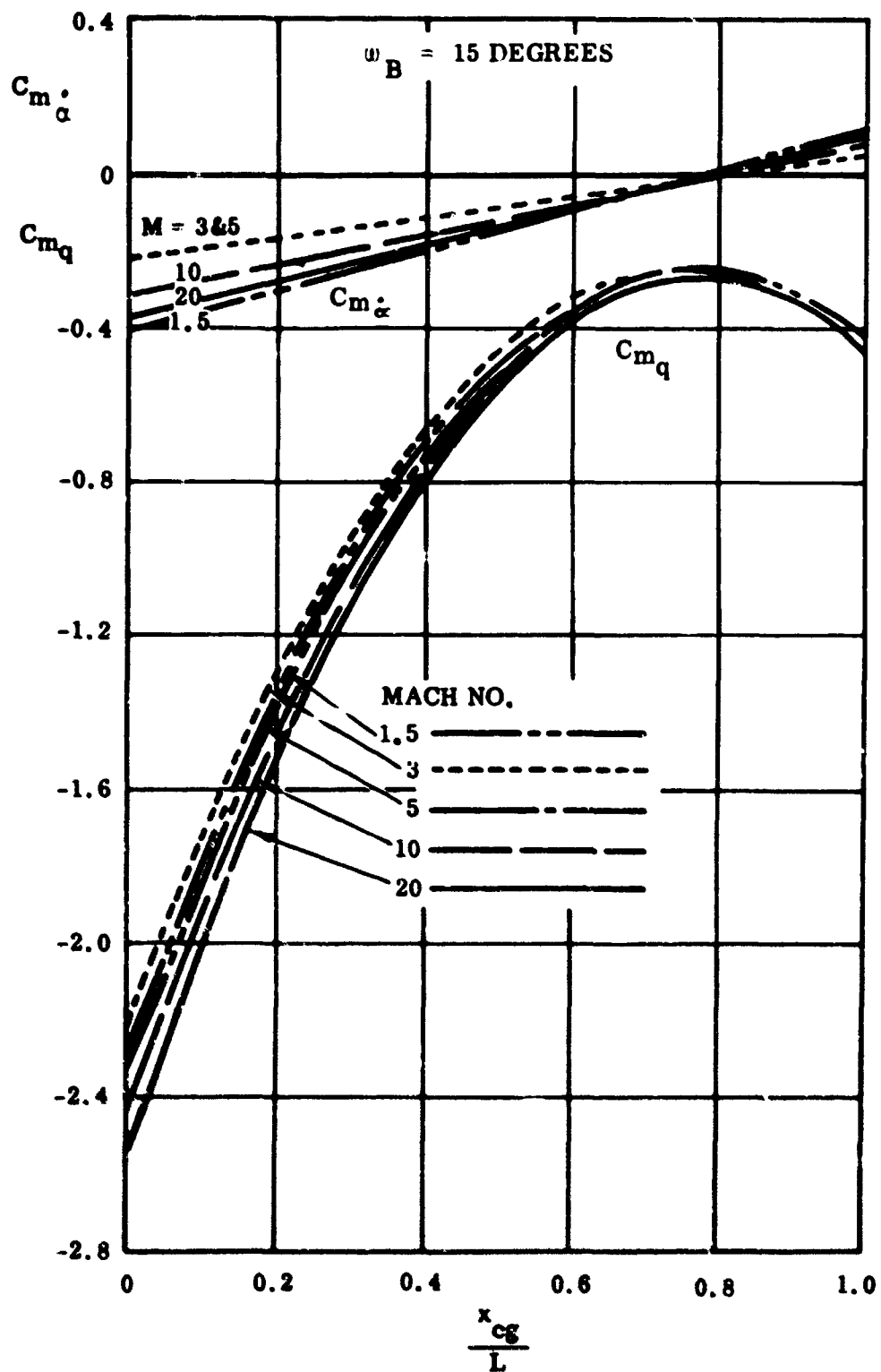Figure 24. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$

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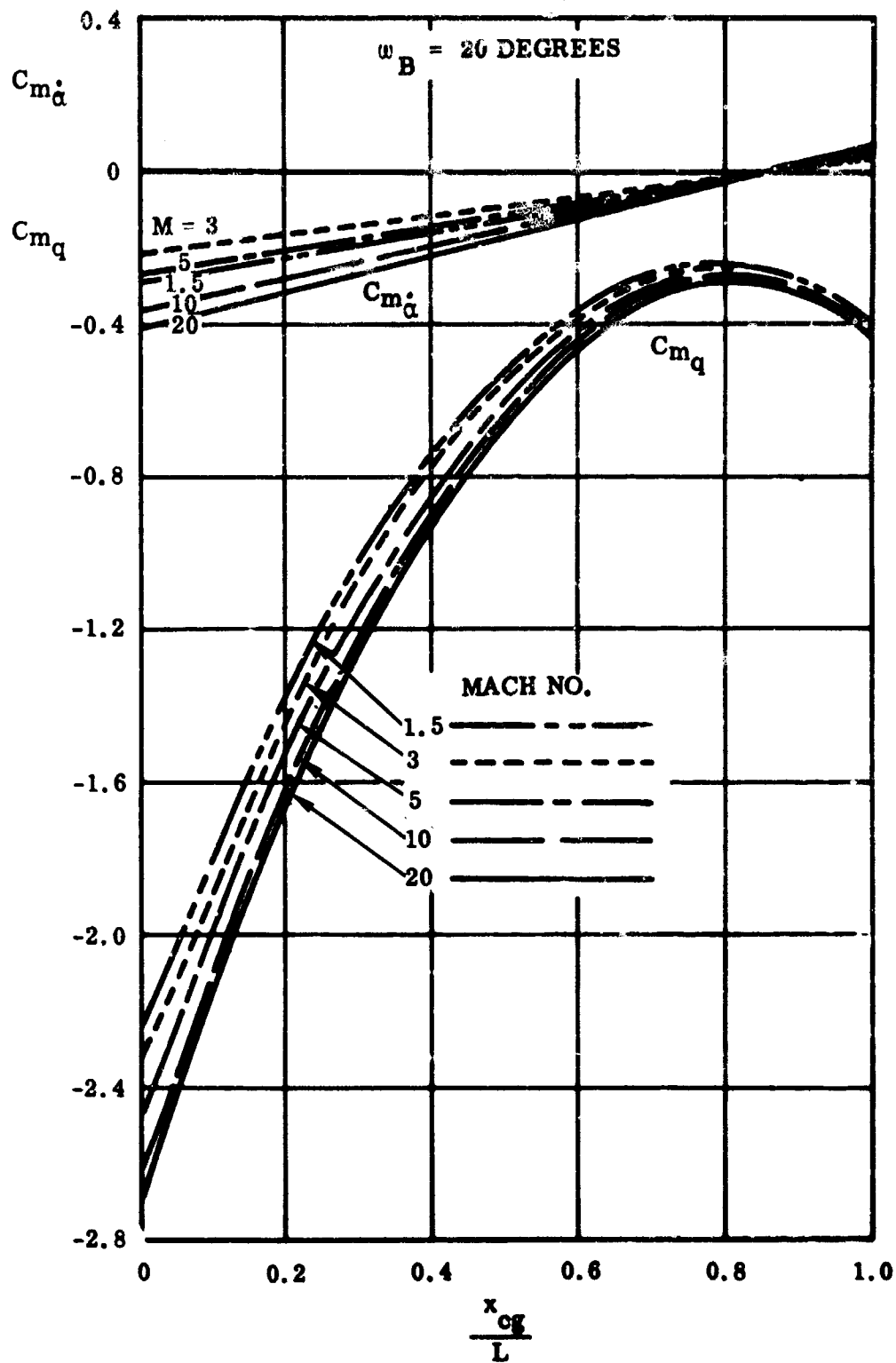Figure 25. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$

Figure 26. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot\alpha}}$
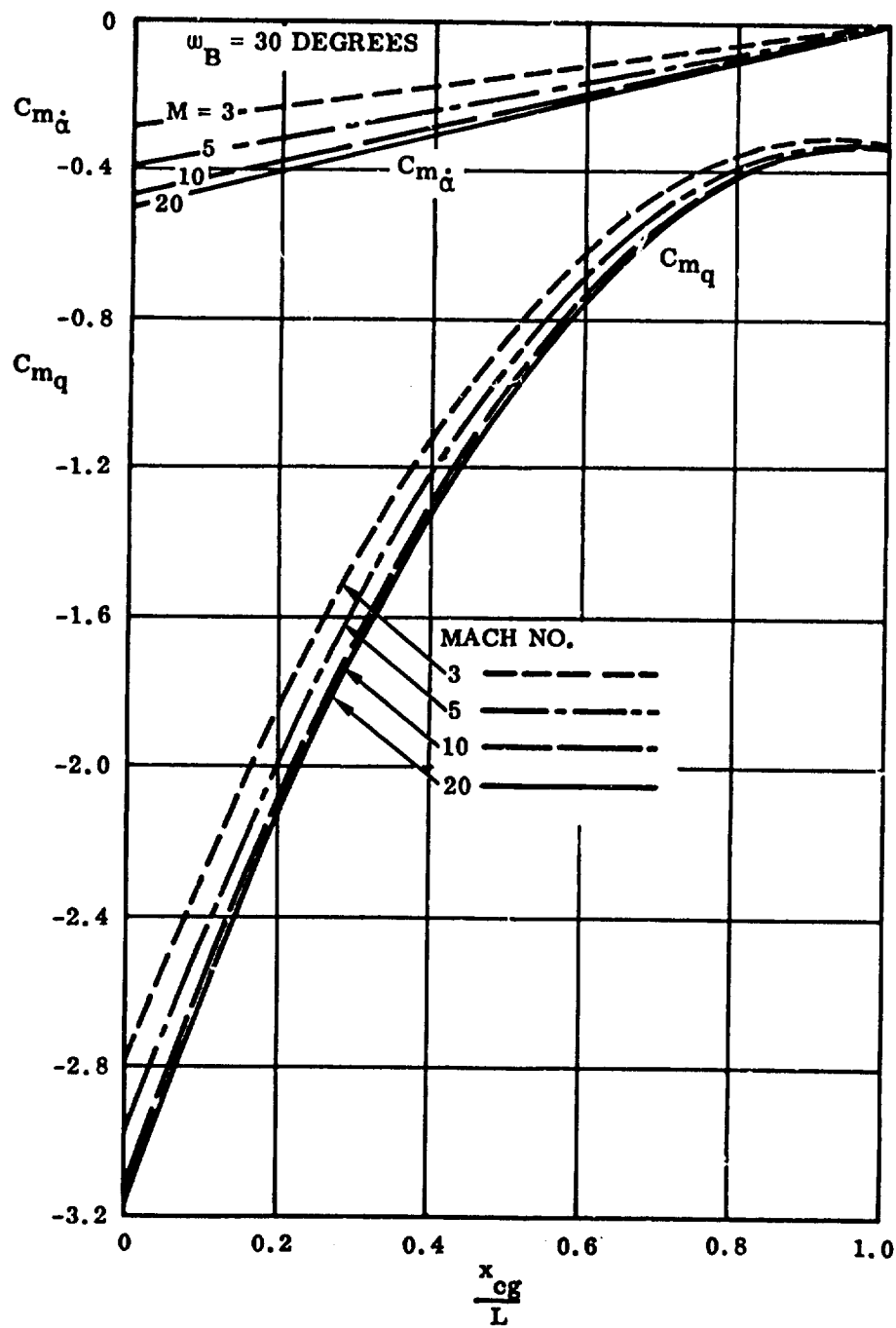
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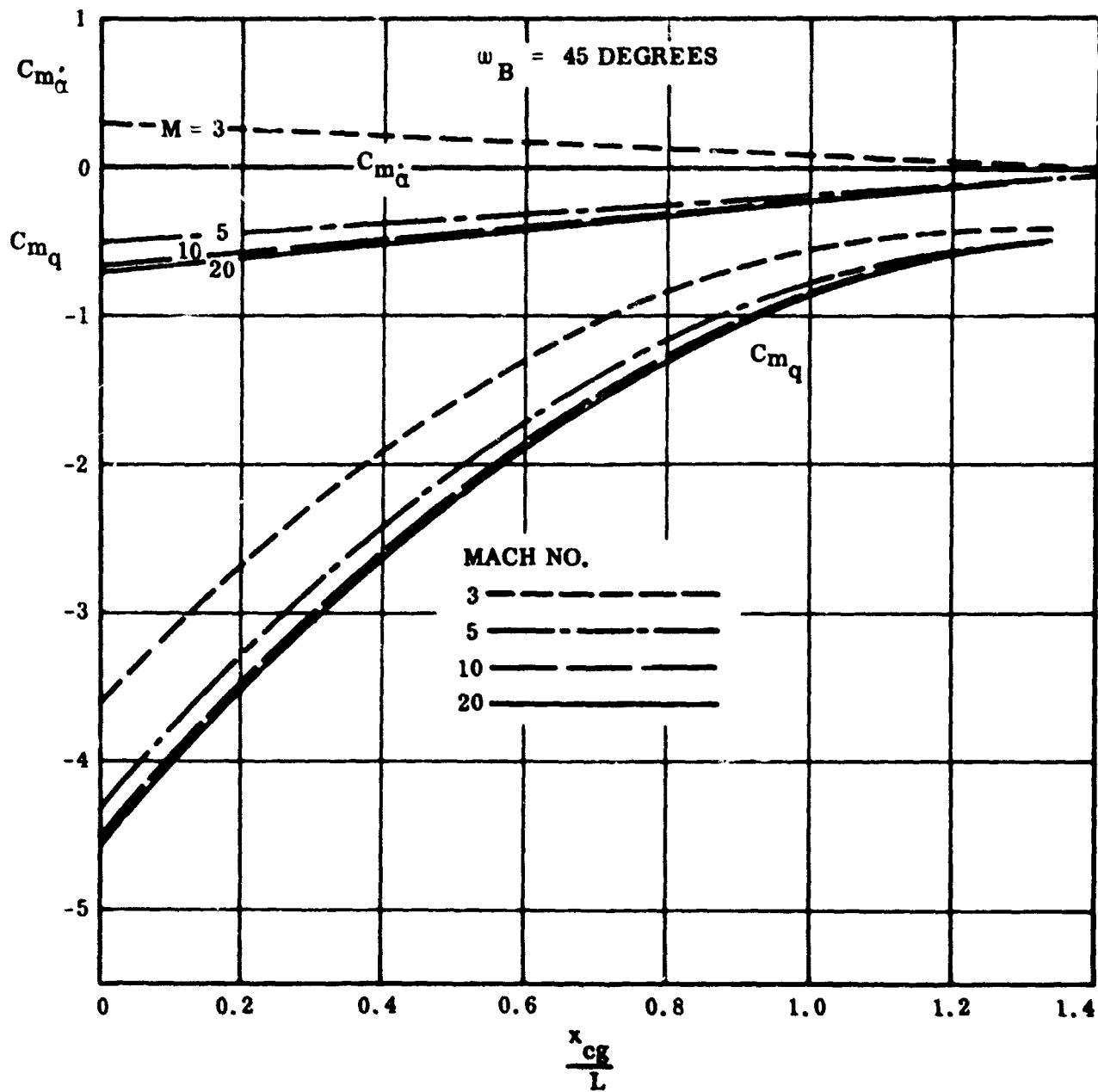Figure 27. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$

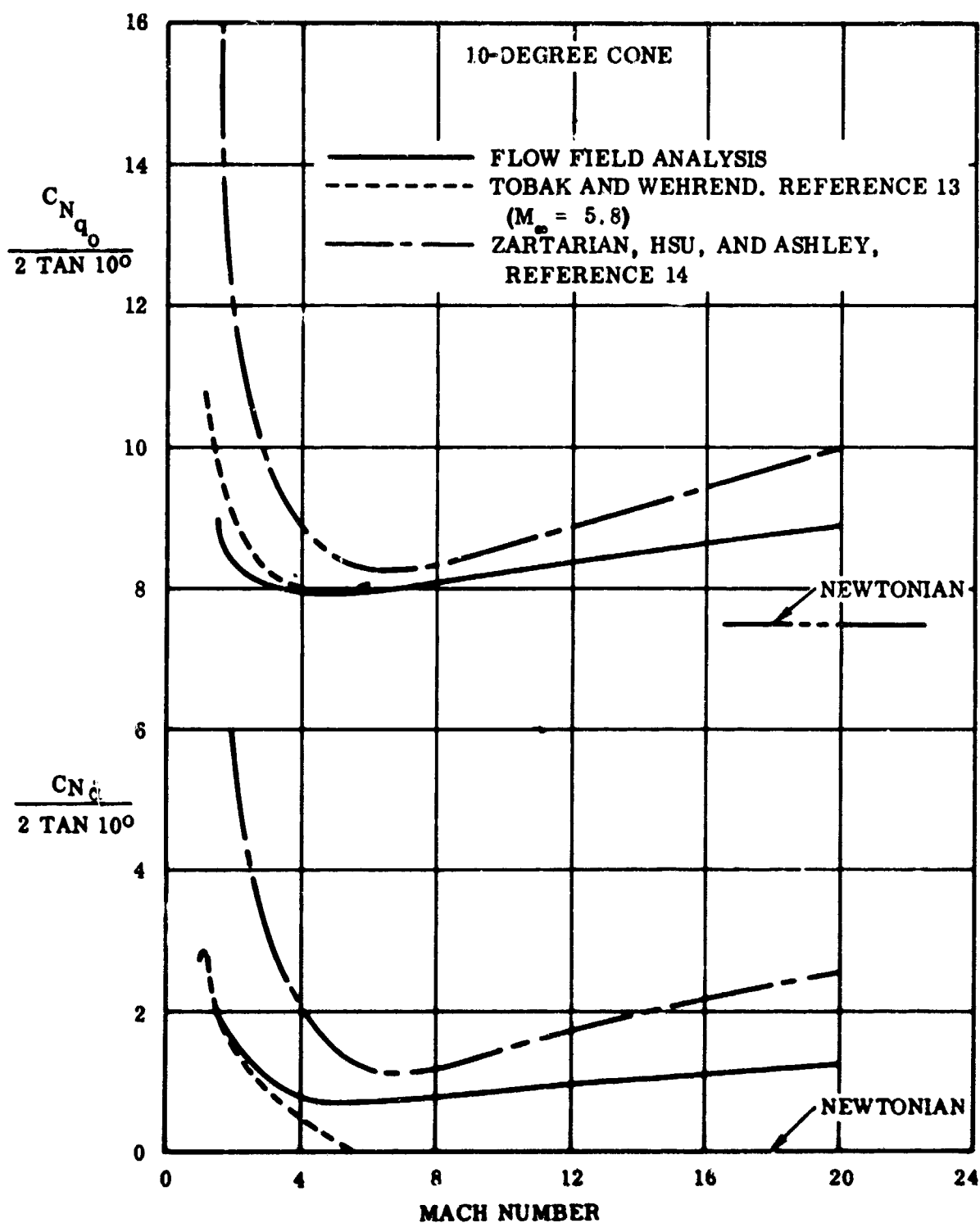Figure 28. Dynamic Moment Derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$

Figure 29. Comparison of Theoretical Results

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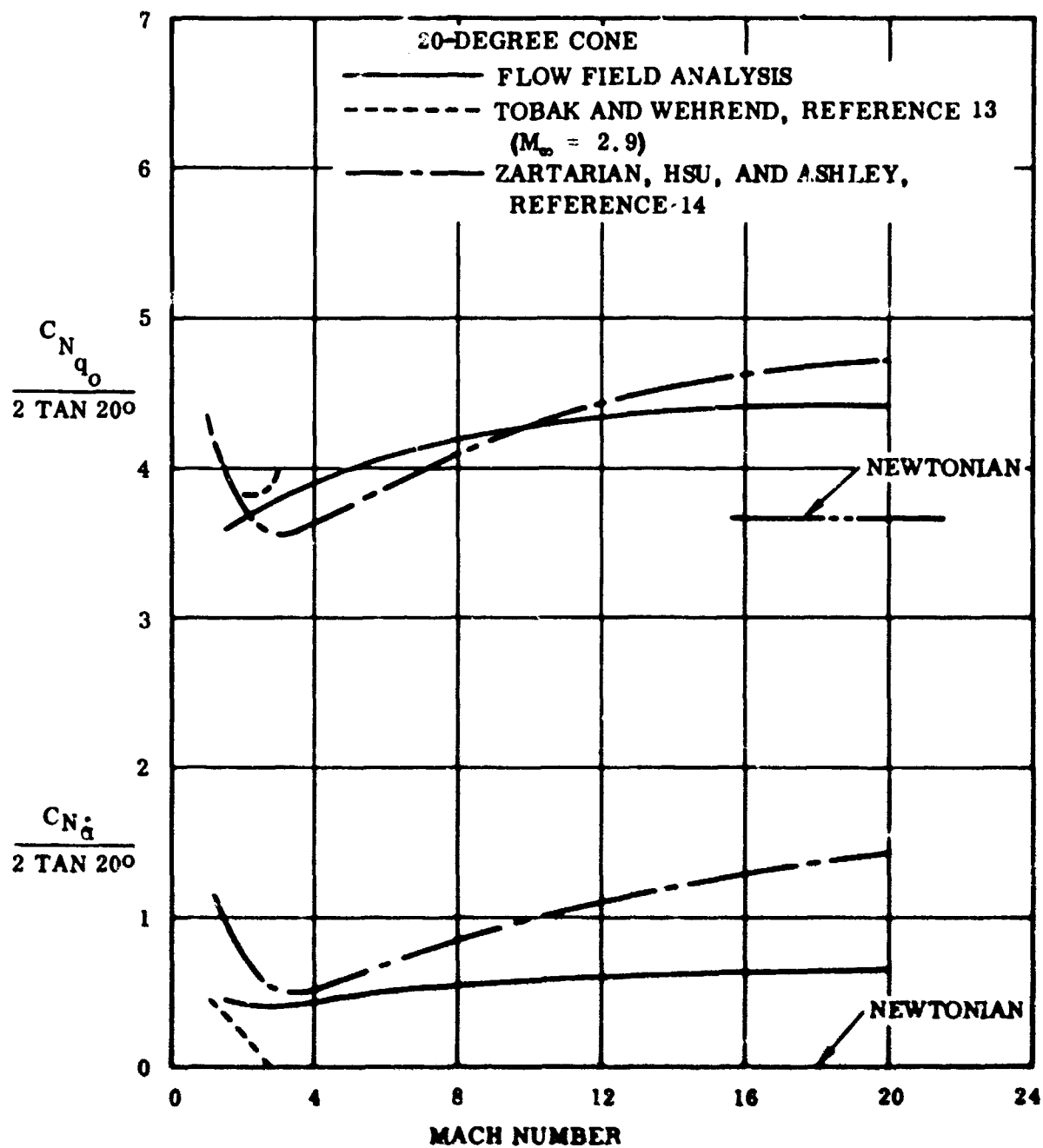
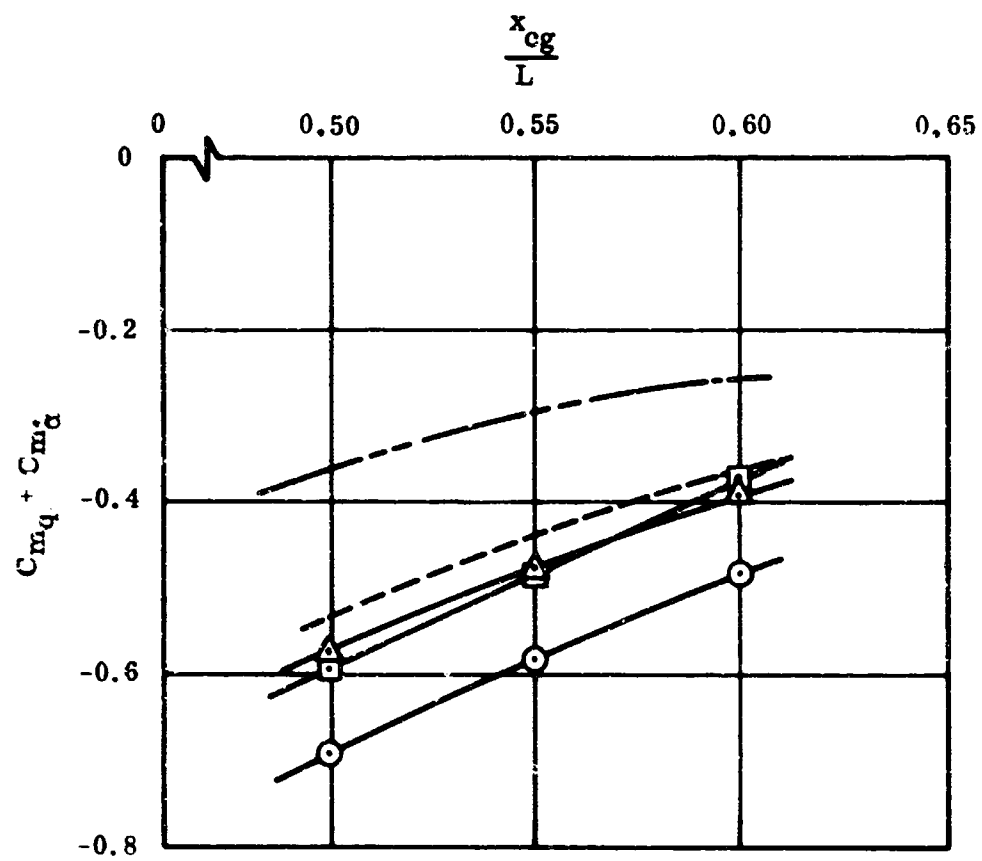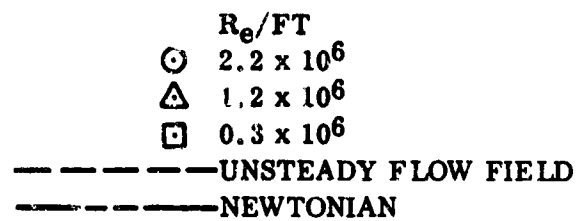Figure 30. Comparison of Theoretical Results

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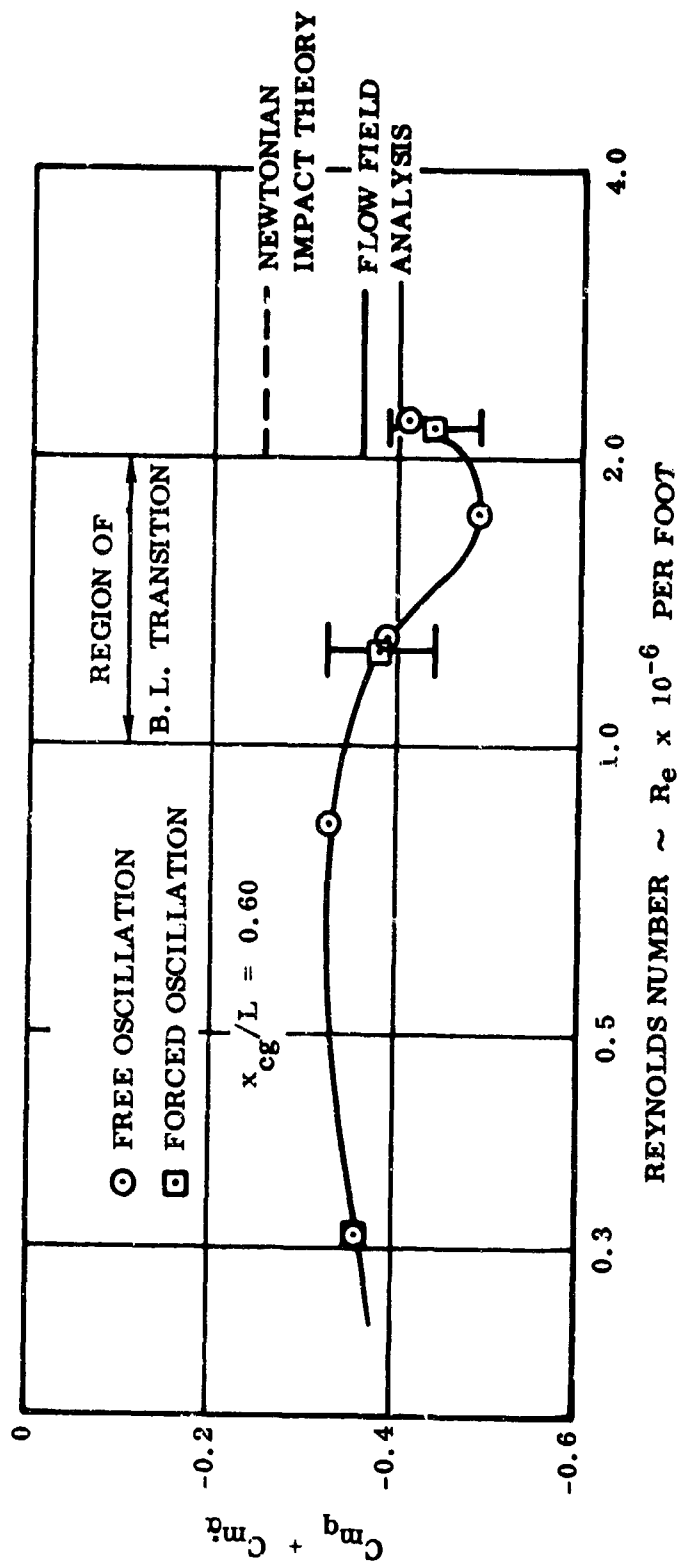Figure 31. Comparison of Theory and Experiment

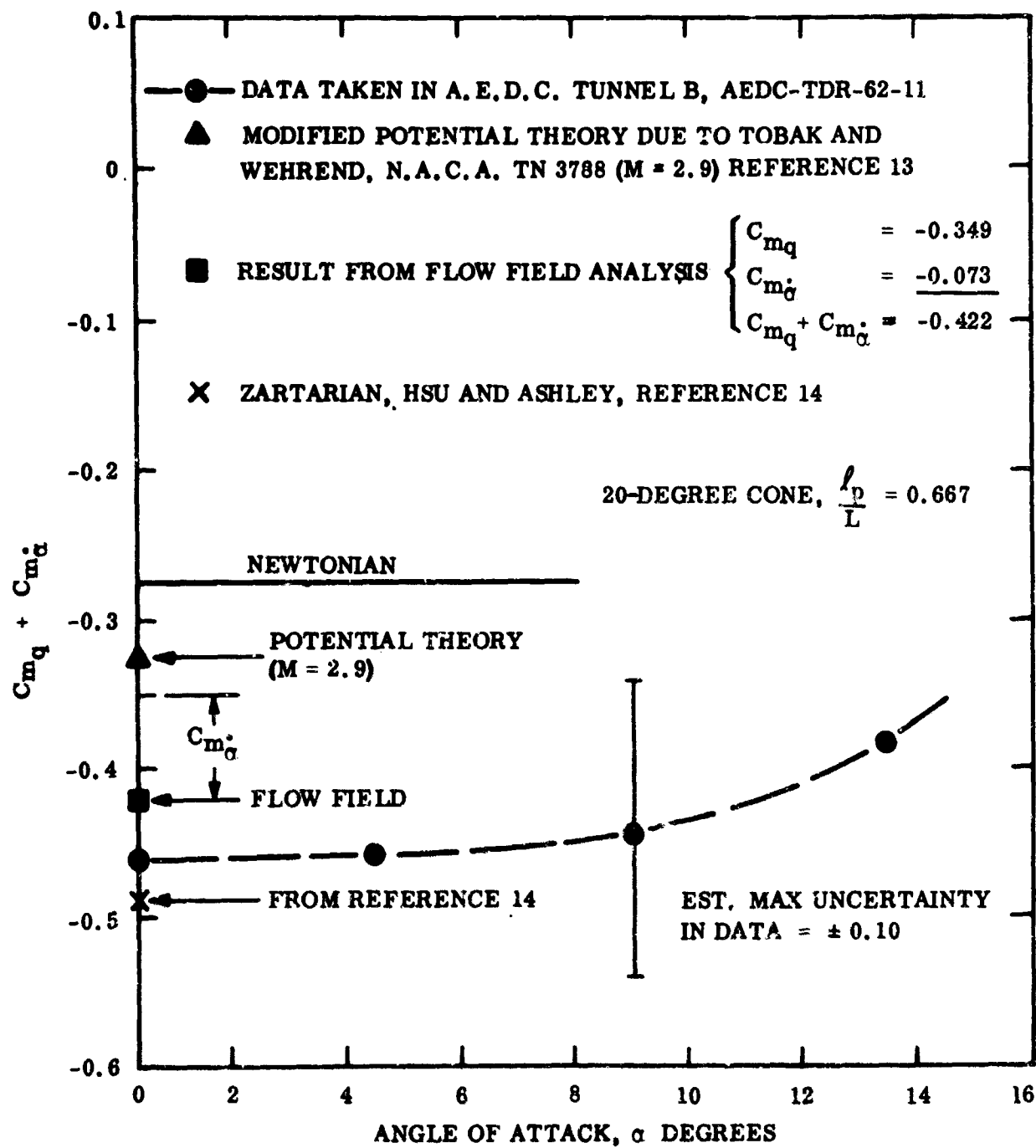Figure 32. Reynolds Number Effects or $C_{m_q} + C_{m_{\dot{\alpha}}}$

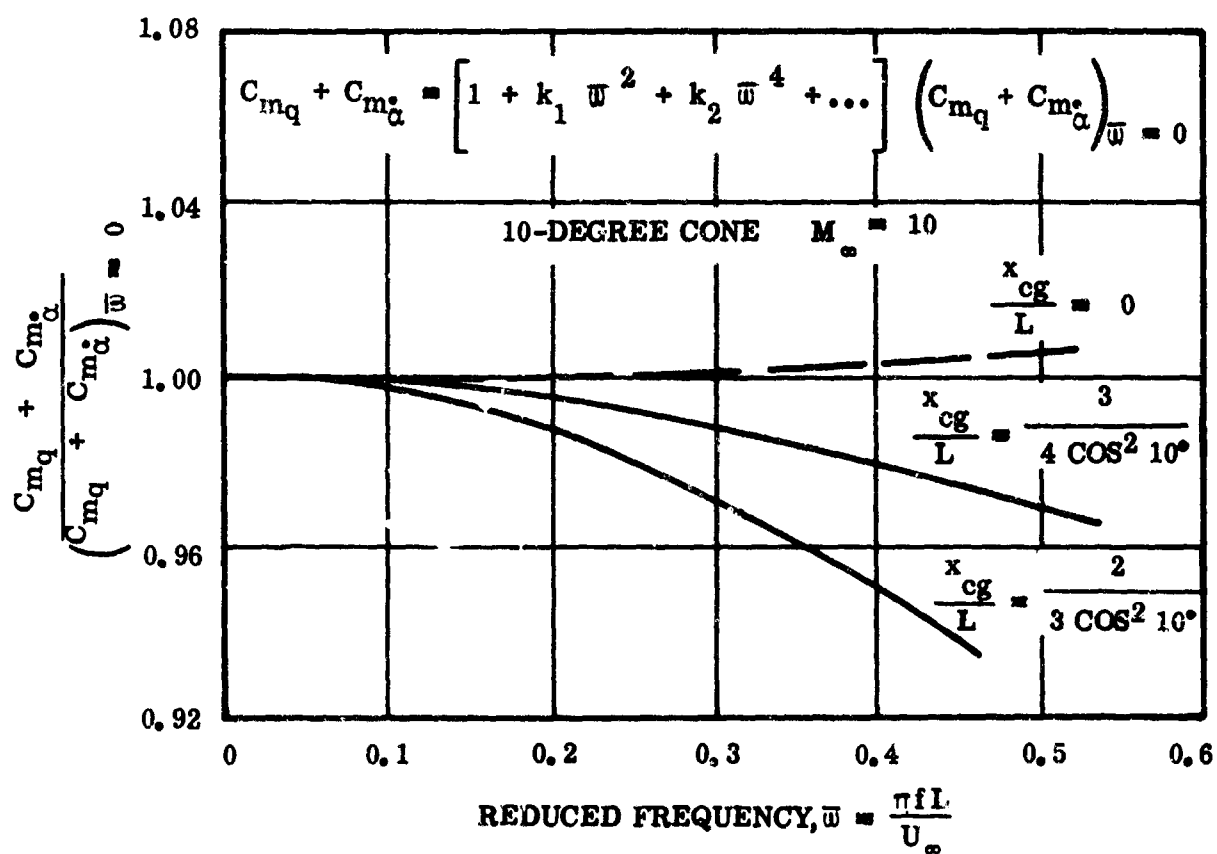Figure 33.   Comparison of Theory and Experiment, M = 8

Figure 34. Theoretical Evaluation of Frequency Effects

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APPENDIX I

SUBSTITUTION FORMS FOR THE PERTURBATION VARIABLES

For $j = 1$

$$p_1 = \sum_{n=0}^{\infty} p_{1,n}(\omega) \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \cos \varphi$$

$$\rho_1 = \sum_{n=0}^{\infty} \rho_{1,n}(\omega) \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \cos \varphi$$

$$S_1 = \sum_{n=0}^{\infty} S_{1,n}(\omega) \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \cos \varphi$$

$$u_1 = \sum_{n=0}^{\infty} u_{1,n}(\omega) \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \cos \varphi$$

$$v_1 = \sum_{n=0}^{\infty} v_{1,n}(\omega) \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \cos \varphi$$

$$w_1 = \sum_{n=0}^{\infty} w_{1,n}(\omega) \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \sin \varphi$$

$$\omega_{s1} = \sum_{n=0}^{\infty} \omega_{1,n} \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{\alpha}}{d t^n} \cos \varphi$$

$(\omega_{1,n}$ are constants)

For j = 2

$$p_2 = \sum_{n=0}^{\infty} \left| p_{2,n}(\omega) \frac{R}{L} - p_{1,n}(\omega) \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{q}}{d t^n} \cos \varphi$$

$$\rho_2 = \sum_{n=0}^{\infty} \left| \rho_{2,n}(\omega) \frac{R}{L} - \rho_{1,n}(\omega) \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{q}}{d t^n} \cos \varphi$$

$$S_2 = \sum_{n=0}^{\infty} \left| S_{2,n}(\omega) \frac{R}{L} - S_{1,n}(\omega) \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{q}}{d t^n} \cos \varphi$$

$$u_2 = \sum_{n=0}^{\infty} \left| u_{2,n}(\omega) \frac{R}{L} - u_{1,n}(\omega) \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{q}}{d t^n} \cos \varphi$$

$$v_2 = \sum_{n=0}^{\infty} \left| v_{2,n}(\omega) \frac{R}{L} - v_{1,n}(\omega) \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{q}}{d t^n} \cos \varphi$$

$$w_2 = \sum_{n=0}^{\infty} \left| w_{2,n}(\omega) \frac{R}{L} - w_{1,n}(\omega) \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty}\right)^n \frac{d^n \bar{q}}{d t^n} \sin \varphi$$

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$$\omega_{s2} = \sum_{n=0}^{\infty} \left| \omega_{2,n} \frac{R}{L} - \omega_{1,n} \frac{x_{cg}}{L} \right| \left(\frac{R}{U_\infty} \right)^n \frac{d^n \bar{q}}{d t^n} \cos \varphi$$

$(\omega_{2,n}$ are constants)

APPENDIX II

EQUATIONS FOR THE FIELD AT ZERO YAW

The following equations determine the unperturbed field at zero yaw:

the continuity Equation,

$$\frac{d \, \rho_0 \, v_0 \, \sin \omega}{d \omega} + 2 \, \rho_0 \, u_0 \, \sin \omega = 0;$$

the condition of irrotationality,

$$\frac{d \, u_0}{d \omega} - v_0 = 0;$$

the state Equation,

$$p_0 = p \, (\rho_0 \, S_0);$$

the integrated energy Equation,

$$S_0 = \text{constant}$$

and, the Bernoulli Equation

$$h \, (\rho, s) + \frac{u^2 + v^2}{2} = \text{const.}$$

The first of these equations can be re-written:

$$v_0 \, \frac{d}{d \omega} \left\{ \sqrt{-\rho_0 \, v_0 \, \sin \omega} \right\} + u_0 \left\{ \sqrt{-\rho_0 \, v_0 \, \sin \omega} \right\} = 0$$

and is used in deriving Equations (5.1), (5.4), and (5.6). These equations are used to replace derivatives of the flow field quantities at zero yaw by equivalent expressions in terms of the flow field variables themselves.

APPENDIX III

EXPRESSIONS FOR THE NORMAL FORCE AND MOMENT

Integration of the perturbation description of the pressures on the surface of the cone gives the following expressions for the normal force (negative of the force in the y direction) and the moment in the z direction about the center of gravity.

$$
C_N = \frac{-F_y}{q_\infty A_B} = \sum_{n=0}^{\infty} N_{1,n} \left(\frac{D}{2 U_\infty}\right)^n \frac{d^n \alpha}{d t^n} + \sum_{n=0}^{\infty} \left\{ N_{2,n} \right.
$$

$$
\left. - \frac{x_{cg}}{L} \frac{N_{1,n}}{\tan \omega_B} \right\} \left(\frac{D}{2 U_\infty}\right)^{n+1} \frac{d^n q}{d t^n}
$$

$$
C_m = \frac{M_z}{q_\infty A_B L} = - \sum_{n=0}^{\infty} \frac{N_{1,n}}{\cos^2 \omega_B} \frac{(n+2)}{(n+3)} \left(\frac{D}{2 U_\infty}\right)^n \frac{d^n \alpha}{d t^n}
$$

$$
+ \sum_{n=0}^{\infty} \left\{ - N_{2,n} \frac{(n+3)}{(n+4)} \right.
$$

$$
\left. + \frac{x_{cg}}{L} \frac{N_{1,n}}{\tan \omega_B} \frac{(n+2)}{(n+3)} \right\} \frac{1}{\cos^2 \omega_B} \left(\frac{D}{2 U_\infty}\right)^{n+1} \frac{d^n q}{d t^n} + \frac{x_{cg}}{L} C_N
$$

where:

$$N_{1,n} = \frac{p_{1,n}}{q_\infty} \qquad \frac{\cos \omega_B}{(n+2) \sin^{n+1} \omega_B}$$

$$N_{2,n} = \frac{p_{2,n}}{q_\infty} \qquad \frac{\cos \omega_B}{(n+3) \sin^{n+2} \omega_B}$$

The normal force derivatives defined in the symbols section are related to the $N_{j,n}$ as follows:

$$\left(\frac{2 U_\infty}{L}\right)^n \frac{\partial C_N}{\partial \left(\dfrac{d^n \alpha}{d t^n}\right)} = (2 \tan \omega_B)^n \ N_{1,n}$$

$$\left(\frac{2 U_\infty}{L}\right)^{n+1} \frac{\partial C_N}{\partial \left(\dfrac{d^n q}{d t^n}\right)} = (2 \tan \omega_B)^{n+1} \ N_{2,n}$$

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13 ABSTRACT

The inviscid flow field about a right circular cone in unsteady planar flight is analyzed by a perturbation technique which is an extension of Stone's treatment of the cone at small yaw. A solution is found in the form of infinite series in the time rates of change of the pitch rate and angle of attack. The linear stability derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$ as well as "higher order" stability derivatives such as $C_{m_{\dot{q}}}$ and $C_{\alpha_{\ddot{G}}}$ are presented for a wide range of cone angles and Mach numbers.

The stability derivatives, C_{m_q} and $C_{m_{\dot{\alpha}}}$, as obtained from this solution are compared to results obtained from second order potential theory, Newtonian impact theory, and an unsteady flow theory due to Zartarian, Hsu, and Ashley. Both the potential theory and the impact theory predict that $C_{m_{\dot{\alpha}}}$ rapidly approaches zero at high Mach numbers while the present theory indicates that $C_{m_{\dot{\alpha}}}$ approaches a value which is on the order of 10 precent to 20 percent of C_{m_q}.

Numerical results obtained from the present theory are also compared to ground-test data on $C_{m_q} + C_{m_{\dot{\alpha}}}$. The agreement is found to be generally good, although the data in some instances indicate a pronounced Reynolds number effect.

The numerical results for the "higher order" coefficients are used to predict the effect of reduced frequency on the parameter $C_{m_q} + C_{m_{\dot{\alpha}}}$ as obtained by the forced oscillation testing techniques. It is found that the predicted effect is very small over the range of reduced frequencies likely to be encountered.

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